Problem for the week of January 30, 2012

Let A and B be $m \times n$ and $p \times n$ matrices, respectively. If $\text{rank}A + \text{rank}B < n$, show that there exists a nonzero n-dimensional vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$.

Solution

證明 $N(A) \cap N(B) \neq \{\mathbf{0}\}$ 即證得原命題。考慮零空間 $N(A)$ 和 $N(B)$ 的容斥關係,

$$\dim(N(A) + N(B)) = \dim N(A) + \dim N(B) - \dim(N(A) \cap N(B))$$

利用秩–零度定理, $\text{rank}A + \dim N(A) = n$, $\text{rank}B + \dim N(B) = n$, 即有

$$\dim(N(A) \cap N(B)) = n - \text{rank}A + n - \text{rank}B - \dim(N(A) + N(B))$$

使用已知條件 $\text{rank}A + \text{rank}B < n$, 而且 $N(A) + N(B)$ 爲 $\mathbb{C}^n$ 的子空間, 可以推論 $\dim(N(A) \cap N(B)) > n - \dim(N(A) + N(B)) \geq 0$. $\square$