Problem for the week of February 13, 2012

An $n \times n$ Householder matrix H has the form $H = I - 2\mathbf{u}\mathbf{u}^T$, where $\|\mathbf{u}\| = 1$. Let $\mathbf{x}, \mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. If $\|\mathbf{x}\| = \|\mathbf{y}\|$, show that there exists a Householder matrix H such that $H\mathbf{x} = \mathbf{y}$.

Solution

令 $H = I - 2\mathbf{u}\mathbf{u}^T$, 其中 $\|\mathbf{u}\| = 1$, 且 $H\mathbf{x} = \mathbf{y}$, 也就有

$$H\mathbf{x} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{x} = \mathbf{x} - 2(\mathbf{u}^T\mathbf{x})\mathbf{u} = \mathbf{y}$$

亦即

$$2(\mathbf{u}^T\mathbf{x})\mathbf{u} = \mathbf{x} - \mathbf{y}$$

觀察出

$$(\mathbf{x} - \mathbf{y})^T(\mathbf{x} + \mathbf{y}) = \mathbf{x}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} - \mathbf{y}^T\mathbf{x} - \mathbf{y}^T\mathbf{y} = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = 0$$

若 $\mathbf{x} \neq \mathbf{y}$, 令 $\mathbf{u} = \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$, 改寫 $\mathbf{x} = \frac{1}{2}[(\mathbf{x} - \mathbf{y}) + (\mathbf{x} + \mathbf{y})]$, 乘開確認

$$2(\mathbf{u}^T\mathbf{x})\mathbf{u} = \frac{(\mathbf{x} - \mathbf{y})^T}{\|\mathbf{x} - \mathbf{y}\|}[(\mathbf{x} - \mathbf{y}) + (\mathbf{x} + \mathbf{y})]\mathbf{u} = \|\mathbf{x} - \mathbf{y}\|\mathbf{u} = \mathbf{x} - \mathbf{y}$$

若 $\mathbf{x} = \mathbf{y}$, 則任意選擇單位長向量 $\mathbf{u} \in \mathrm{span}\{\mathbf{x}^\perp\}$ 使得 $\mathbf{u}^T\mathbf{x} = 0$。 $\square$