Problem for the week of February 20, 2012

Find the determinant of the following $2n \times 2n$ matrix:

$$A=\begin{bmatrix} 1 & 2 & 1 & 1 & \cdots & 1 & 1\\ 2 & 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & 1 & 2 & \cdots & 1 & 1\\ 1 & 1 & 2 & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 1 & 1 & \cdots & 1 & 2\\ 1 & 1 & 1 & 1 & \cdots & 2 & 1 \end{bmatrix}.$$

Solution

矩陣 A 可分解如下:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}+\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & 1 & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}=B+E$$

其中 $E=\mathbf{e}\mathbf{e}^T$, $2n$ 維向量 $\mathbf{e}$ 的所有元皆爲 1。利用矩陣行列式引理 (matrix determinant lemma),

$$\det A=\det(B+\mathbf{e}\mathbf{e}^T)=\det B+\mathbf{e}^T(\mathrm{adj}B)\mathbf{e}=\det B+\sum_{i=1}^{2n}\sum_{j=1}^{2n}(\mathrm{adj}B)_{ij}$$

矩陣 B 的行列式爲其主對角分塊行列式乘積, 就有

$$\det B=\begin{vmatrix} 0 & 1\\ 1 & 0 \end{vmatrix}^n=(-1)^n$$

伴隨矩陣 $\mathrm{adj}B$ 的非零元僅有 $(\mathrm{adj}B)_{2k,2k-1}=(\mathrm{adj}B)_{2k-1,2k}=(-1)^n$, $k=1,\ldots,n$。合併以上結果, 即得 $\det A=(-1)^n+2n(-1)^n=(-1)^n(2n+1)$。

另一個做法是直接利用行列式基本性質計算化簡。觀察出矩陣 A 的所有列總和皆爲 $2n+1$, 故可將第 $2,3,\ldots,2n$ 行加至第 1 行, 再將所得矩陣的第 $2,3,\ldots,2n$ 列各自減去第 1 列, 如下:

$$\det A = \begin{vmatrix} 2n+1 & 2 & 1 & 1 & \cdots & 1 & 1 \\ 2n+1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 2n+1 & 1 & 1 & 2 & \cdots & 1 & 1 \\ 2n+1 & 1 & 2 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2n+1 & 1 & 1 & 1 & \cdots & 1 & 2 \\ 2n+1 & 1 & 1 & 1 & \cdots & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2n+1 & 2 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

至此即可對第 1 行展開, 再對 $(2n-1)\times(2n-1)$ 階新矩陣的第 1 列展開:

$$\det A = (2n+1)\begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix} = (2n+1)(-1)\begin{vmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

如前述做法, $(2n-2)\times(2n-2)$ 階矩陣的行列式爲其主對角分塊行列式乘積, 於是得到

$$\det A = (2n+1)(-1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}^{n-1} = (2n+1)(-1)(-1)^{n-1} = (-1)^n(2n+1)$$

□