Problem for the week of February 6, 2012

Let $M_2(\mathbb{R})$ be the vector space of all 2×2 real matrices. The standard basis for $M_2(\mathbb{R})$ is given as follows:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Define the linear transformation

$$T(X) = XA - AX$$

where $X \in M_2(\mathbb{R})$, and $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) Find the matrix representation of T under the basis E_i, $i = 1, 2, 3, 4$.

(b) Find a basis for the range of T.

(c) Find a basis for the kernel of T.

Solution

(a) 令 $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, 展開 $T(X)$, 如下:

$$T(X) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2c & 2a - 2d \\ 0 & 2c \end{bmatrix}$$

將標準基底 E_i 代入計算 $T(E_i)$, 可得

$$T(E_1) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},\ T(E_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ T(E_3) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix},\ T(E_4) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

所以線性變換 T 參考標準基底的表示矩陣爲

$$[T] = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(b) 對表示矩陣 $[T]$ 執行基本列運算, 可得下列簡化列梯形矩陣:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

這顯示 $[T]$ 的第 1 和 3 行為軸行, 故 T 的值域基底可為 $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$。

(c) 由 (b) 可知 $[T]$ 的零空間由 $(0,1,0,0)^T$ 和 $(1,0,0,1)^T$ 擴張而成, 故 T 的核基底可為 $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$。 $\square$