**Linear Algebra** **Final Exam**

6/20/2005

**You must show your work to receive credit.**

1. (20%) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

   (a) (4%) Give an orthonormal basis for the column space of $A$.
   (b) (3%) Find all vectors that are perpendicular (orthogonal) to $C(A)$.
   (c) (4%) Find the projection matrix $P$ projecting onto the column space of $A$.
   (d) (3%) What is the rank of $P$?
   (e) (3%) Is $P$ diagonalizable? Why?
   (f) (3%) Is it true that $C(P)=C(A)$? Justify your answer.

2. (12%) True or false (give a reason if true or a counterexample if false). The following matrices are all square.
   (a) (3%) The determinant of $A+B$ equals det $A$ + det $B$.
   (b) (3%) The determinant of $\mathbf{u}\mathbf{u}^T$ must be zero, where $\mathbf{u}$ is a vector in $R^3$.
   (c) (3%) det $A$ = det $R$, where $R$=rref($A$).
   (d) (3%) If $A$ and $B$ are invertible, then $AB$ is invertible.

3. (17%) Consider the following sequence: $F_0$=0, $F_1$=1 and $F_{k+2} = (2F_{k+1} + cF_k)/3$. We can write the matrix equation

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = A \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 2/3 & c/3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}.$$

   (a) (3%) Which choices of $c$ give orthogonal eigenvectors of $A$?
   (b) (3%) Which choices of $c$ won't give independent eigenvectors (of $A$)?
   (c) (5%) If $c$=1, write $A$ as a diagonalized form: $A = S\Lambda S^{-1}$, where $\Lambda$ is a diagonal eigenvalue matrix and $S$ is an eigenvector matrix.
   (d) (3%) If $c$=1, find the limit of $F_k$ as $k \to \infty$.
   (e) (3%) If $c$= −1, will it be true that $F_k \to 0$ as $k \to \infty$? State your reason.

4. (10%) If 3 by 3 matrix $A$ is known to have eigenvalues 1, 1, 2. Find the following.
   (a) (3%) the rank of $A$.
   (b) (3%) the eigenvalues of $(A+I)^{-1}$.
   (c) (4%) the determinant of $A-I$.

5. (11%) Consider the following system of differential equations:

$$\begin{bmatrix} \dfrac{dy_1}{dt} \\ \dfrac{dy_2}{dt} \end{bmatrix} = A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(a) (8%) If $y_1(0)=1$ and $y_2(0)=6$, solve the system. That is, what are values of $y_1$ and $y_2$ at $t$?

(b) (3%) What is $e^A$ ? Show this 2 by 2 matrix.

6. (20%) True or false (give a reason if true or a counterexample if false). The following matrices are all square.

(a) (4%) If $A$ is symmetric positive definite and $A$ is similar to $B$, then $B$ is symmetric positive definite.

(b) (4%) $A$ and $A^T$ have the same eigenvalues and their corresponding eigenvectors are the same.

(c) (4%) $(A+B)^2 = A^2 + 2AB + B^2$.

(d) (4%) If $A$ is symmetric positive definite, then $A^{-1}$ is symmetric positive definite.

(e) (4%) trace($AB$)=trace($BA$).

7. (10%) Consider the following linear transformation problem in $R^2$. The $v$-basis is given by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The $w$-basis is $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(a) (3%) Find the matrix $M$ that maps the coordinates in $v$-basis to the coordinates in $w$-basis, i.e., $[\mathbf{x}]_w = M[\mathbf{x}]_v$.

(b) (3%) Find the matrix $N$ that maps the coordinates in $w$-basis to the coordinates in $v$-basis, i.e., $[\mathbf{x}]_v = N[\mathbf{x}]_w$.

(c) (4%) If $A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$ is the matrix for a linear transformation $T$ in $v$-basis, what is the corresponding matrix for the same transformation in $w$-basis?