Linear Algebra **Final Exam**

6/26/2007

You must show your work to receive credit.

1. (23%) Let V be the subspace of $\boldsymbol{R}^3$ consisting of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + 2y - z = 0$.

   (a) (3%) Find all vectors that are perpendicular (orthogonal) to V.

   (b) (4%) Find a 3×2 matrix A whose column space is V, i.e., C(A)=V.

   (c) (4%) Find an orthonormal basis for V.

   (d) (5%) Find the projection matrix P projecting onto the left nullspace (not column space) of A, i.e., $N(A^T)$.

   (e) (3%) What is the rank of P in part (d)?

   (f) (4%) Is it true that $C(P)$ is orthogonal to $C(A)$? State your reasoning.

2. (21%) In each case give all the information you can about the eigenvalues and eigenvectors, when the n×n real matrix A has the following property:

   (a) (3%) A is not diagonalizable.

   (b) (3%) $\det(A^2)=0$.

   (c) (3%) A is symmetric positive definite.

   (d) (3%) The eigenvalues of $A+I$ are 1, 2, and 3.

   (e) (3%) The powers A^k approach the zero matrix.

   (f) (3%) rank(A)=n.

   (g) (3%) A is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

3. (16%) Define a sequence of numbers in the following way: F_0=0, F_1=1/2, and $F_{k+2} = (F_{k+1} + F_k)/2$.

   (a) (3%) Set up a 2×2 matrix A to get from $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ to $\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$.

   (b) (5%) Write A as a diagonalized form: $A = S\Lambda S^{-1}$, where Λ is a diagonal eigenvalue matrix and S is an eigenvector matrix.

   (c) (5%) Find the formula for F_k.

   (d) (3%) Find the limit of F_k as $k \to \infty$.

4. (10%) Consider the following system of differential equations:

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(a) (6%) If $y_1(0)=1$ and $y_2(0)=0$, solve the system. That is, what are values of y_1 and y_2 at t?

(b) (4%) Find the limit of $y_1(t)$ and $y_2(t)$ as $t \to \infty$.

5. (14%) Let the quadratic form be $Q(\mathbf{x}) = x_1^2 + 10x_1x_2 + x_2^2$.

(a) (3%) Set up a 2×2 symmetric matrix A so that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

(b) (5%) Orthogonally diagonalize A as $P\Lambda P^T$, where $P^T = P^{-1}$.

(c) (6%) Use the results in part (b) to find the unit vector $\mathbf{x}$ at which $Q(\mathbf{x})$ is maximized, subject to the constraint $\|\mathbf{x}\| = 1$. What is the maximum value of $Q(\mathbf{x})$, subject to $\|\mathbf{x}\| = 1$?

6. (14%) Let T: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the linear transformation satisfying $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

(a) (4%) Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) (4%) What is the matrix A expressing T in terms of the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

(c) (6%) What is the matrix B expressing T in terms of the following basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?