Linear Algebra **Final Exam**

6/17/2008

1. (20%) Let V be the subspace of $\mathbf{R}^3$ consisting of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

   (a) (3%) Find all vectors that are perpendicular (orthogonal) to V.
   (b) (4%) Find a matrix A whose column space is V, i.e., $C(A)$=V.
   (c) (4%) What is the projection matrix P projecting onto V?
   (d) (4%) Is the projection matrix P in (c) positive definite or semi-positive definite? State your reasoning.
   (e) (5%) What is the closest vector in the orthogonal complement $V^{\perp}$ to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$?

2. (28%) Suppose A is a 3 by 3 matrix having eigenvalues $-3,\ 1,\ 2.$ Answer the following questions.
   (a) (3%) What is the rank of A^TA?
   (b) (3%) What is the determinant of 2A?
   (c) (3%) What is the trace of 2A?
   (d) (3%) What are the eigenvalues of A^2?

   Yes or no. State your reasoning.
   (e) (4%) Do the powers A^k converge to the zero matrix?
   (f) (4%) Is $A - I$ invertible? I is the 3 by 3 identity matrix.
   (g) (4%) Is A similar to $B = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$?
   (h) (4%) Is A always symmetric?

3. (17%) Consider the following bases for $\mathbf{R}^2$.

   The v-basis is given by $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The w-basis is given by $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) (4%) Find the coordinate vector of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ with respect to w-basis.

(b) (5%) Suppose the matrix M maps the coordinate vector in v-basis to the coordinate vector in w-basis, i.e., $[\mathbf{x}]_w = M[\mathbf{x}]_v$. Find M.

(c) (4%) Suppose A is the matrix for a linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ in v-basis, and B is the corresponding matrix for the same transformation in w-basis. What is the relationship between A and B? Describe it with one equation.

(d) (4%) In part(c), are you sure that A and B have the same set of eigenvalues? State your reasoning.

4. (20%) Consider the following differential equations:

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & c \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(a) (5%) If c=0, write down the exponential e^A.

(b) (5%) Describe the condition of c so that A can't be diagonalized.

(c) (5%) If c=0, $x(0)$=1 and $y(0)$=1, what are the values of x and y at time t?

(d) (5%) Describe the condition of c so that the above system is always stable.

5. (15%) Consider the following quadratic form $\mathbf{x}^T A\mathbf{x}$, where A is a 2 by 2 symmetric matrix. It is known that the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 1$, and the corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(a) (5%) Determine A.

(b) (5%) Find the unit vector $\mathbf{x}$ ($\|\mathbf{x}\| = 1$) at which $\mathbf{x}^T A\mathbf{x}$ is maximized. What is the maximum value of $\mathbf{x}^T A\mathbf{x}$?

(c) (5%) Find the unit vector $\mathbf{x}$ ($\|\mathbf{x}\| = 1$) at which $\|A\mathbf{x}\|$ is minimized. What is the minimum value of $\|A\mathbf{x}\|$?