Linear Algebra

Solutions to Final Exam 2005

1.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and thus the column space of A is spanned by its first two columns $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$. By Gram-Schmidt process, the basis for C(A) contains two orthonormal vectors: $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

(b) The required vectors are spanned by a basis for $N(A^T)$, which is any vector along $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(c) With the orthogonal vectors in (a), the projection matrix onto $C(A)$ can be computed by $P = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

(d) rank(P)=rank(A)=dim $C(A)$=2 (see problem (f))

(e) YES. Since P is symmetric, it must be diagonalizable.

(f) YES. $P=A(A^TA)^{-1}A^T$ so that columns of P are linear combinations of columns of A. Also, rank(A)=rank(P).

2.

(a) FALSE. For example, $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det(A+B) = 4 \neq \det A + \det B = 2$.

(b) TRUE. rank($\mathbf{uu}^T$)=1 <3. So, $\mathbf{uu}^T$ is singular and det($\mathbf{uu}^T$)=0.

(c) FALSE. For example, $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \det A = 4, \det R = 1$.

(d) TRUE. Since A and B are invertible, det A and det B are nonzero, so is det AB=(det A)(det B).

3.

(a) If A is symmetric, then it has orthogonal eigenvectors. So, c=3 will do it.

(b) When A has repeated eigenvalues, it may or may not generate independent eigenvectors. When $c=-\frac{1}{3}$, A doesn't have independent eigenvectors.

(c) $A=S\Lambda S^{-1}=\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1/3\end{bmatrix}\frac{1}{4}\begin{bmatrix}3 & 1\\ -1 & 1\end{bmatrix}$. Note that S matrix is not unique.

(d) The solution to the difference equation is

$$\mathbf{u}_k=\frac{1}{4}\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \left(\frac{-1}{3}\right)^k\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix},\quad \text{and thus } F_k=\frac{1}{4}(3-3(-\frac{1}{3})^k).$$

Thus, F_k approaches 3/4 as k approaches infinity.

(e) We require that $|\lambda|<1$ for every eigenvalue of A if F_k approaches 0. If $c=-1$, the eigenvalues of A are $\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$. This confirms that F_k approaches 0 as k becomes infinity.

4.

(a) rank A=3 since all eigenvalues are nonzero.

(b) The eigenvalues of $A+I$ are 2, 2, and 3. So the eigenvalues of its inverse are 1/2, 1/2 and 1/3.

(c) det (A-I)=0 since the eigenvalues of A-I is 0, 0, 1.

5.

(a) Diagonalize A as $A=\begin{bmatrix}1 & 1\\ 1 & -4\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}\frac{1}{5}\begin{bmatrix}4 & 1\\ 1 & -1\end{bmatrix}$. Then, the solution of the differential equation is

$$\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -4\end{bmatrix}\begin{bmatrix}e^{2t} & 0\\ 0 & e^{-3t}\end{bmatrix}\frac{1}{5}\begin{bmatrix}4 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}y_1(0)\\ y_2(0)\end{bmatrix}.$$ Substituting the initial values into the above equation yields $\begin{matrix}y_1=2e^{2t}-e^{-3t}\\ y_2=2e^{2t}+4e^{-3t}\end{matrix}$.

(b) From (a), $e^A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} e^2 & 0 \\ 0 & e^{-3} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$.

6.

(a) FALSE. $B = M^{-1}AM$ and $B^T = M^T A (M^{-1})^T$. To ensure that $B = B^T$, we require that $M^{-1} = M^T$, meaning that M is an orthogonal matrix. In general cases, this is not necessarily true.

(b) FALSE. A and A^T have the same eigenvalues, but the eigenvectors are not necessarily the same.

(c) FALSE. $(A+B)^2 = A^2 + AB + BA + B^2$ but usually $AB \neq BA$.

(d) TRUE. If $A=A^T$, then $A^{-1} = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$, and therefore A^{-1} is also symmetric. Since the eigenvalues of A^{-1} are reverses of eigenvalues of A, A^{-1} is also positive definite.

(e) TRUE. Expanding the matrices AB and BA will prove the relationship, but examples will be suffice to "see" it.

7.

(a) Consider the coordinates of $\mathbf{w}_1$ and $\mathbf{w}_2$ in w-basis and v-basis,

$[\mathbf{w}_1]_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = M[\mathbf{w}_1]_v = M \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[\mathbf{w}_2]_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = M[\mathbf{w}_2]_v = M \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Rewrite them in matrix form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Thus,

$M = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(b) $N = M^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) Let B be the matrix for T in w-basis. Then, $B=MAM^{-1}=\frac{1}{2} \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$.