Linear Algebra
Solutions to Final Exam 2007

1.

(a) V is the nullspace of $B=\begin{bmatrix}1 & 2 & -1\end{bmatrix}$. The orthogonal complement to V is the row space of B, which is spanned by $\begin{bmatrix}1\\2\\-1\end{bmatrix}$.

(b) The nullspace matrix of B is $A=\begin{bmatrix}-2 & 1\\1 & 0\\0 & 1\end{bmatrix}$, satisfying $C(A)=V$.

(c) From (b), let $\mathbf{u}_1=\mathbf{a}_1=\begin{bmatrix}-2\\1\\0\end{bmatrix}$. And, $\mathbf{u}_2=\mathbf{a}_2-\frac{\mathbf{a}_2^T\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}-\frac{-2}{5}\begin{bmatrix}-2\\1\\0\end{bmatrix}=\frac{1}{5}\begin{bmatrix}1\\2\\5\end{bmatrix}$. An orthonormal basis for V is $\left\{\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix},\frac{1}{\sqrt{30}}\begin{bmatrix}1\\2\\5\end{bmatrix}\right\}$.

(d) Note that $N(A^T)=C(A)^{\perp}=V^{\perp}=C(B^T)$. From (a), the projection matrix P is simply $P=\frac{1}{6}\begin{bmatrix}1 & 2 & -1\\2 & 4 & -2\\-1 & -2 & 1\end{bmatrix}$.

(e) rank(P)=1

(f) YES. $C(P)=C(B^T)$ and from (d) it follows that $C(P)=C(A)^{\perp}$.

2.

(a) A has repeated eigenvalues and the corresponding eigenvectors are not independent.

(b) A is singular, and thus at least one of the eigenvalues of A is zero.

(c) A has positive eigenvalues, and A has independent eigenvectors (because A is symmetric).

(d) The eigenvalues of A are 0, 1, and 2, and thus A has independent eigenvectors (because all eigenvalues are distinct).

(e) The absolute value of every eigenvalue is less than 1.

(f) A is invertible, and thus all eigenvalues are nonzero.

(g) A has eigenvalues 1, -1, 2 (two similar matrices have identical eigenvalues). A has independent eigenvectors.

3.

(a) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

(c) $F_k = \frac{1}{3}\left(1 + 2\left(-\frac{1}{2}\right)^{k+1}\right)$

(d) $F_k \to \frac{1}{3}$ as $k \to \infty$

4.

(a) Diagonalize A as $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1 + e^{-2t}\right) \\ \frac{1}{2}\left(1 - e^{-2t}\right) \end{bmatrix}.$$

(b) $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ as $t \to \infty$

5.

(a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

(b) $A = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)$

(c) When $\mathbf{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Q(\mathbf{x})=6$.

6.

(a) $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)-T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\6\end{bmatrix}-\begin{bmatrix}-3\\2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$

(b) Because the two columns of A are $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$, $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$, it follows that $A=\begin{bmatrix}-3 & 5\\2 & 4\end{bmatrix}$.

(c) Let $C=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$. Then, $B=C^{-1}AC=\begin{bmatrix}-5 & -4\\2 & 6\end{bmatrix}$.