Linear Algebra
Solutions to Final Exam 2008

1. V is the nullspace of $B=\begin{bmatrix}1 & 2 & -3\\ 0 & 1 & -2\end{bmatrix}$, i.e., V=$N(B)$.

   (a) The vectors orthogonal to V is the row space of B, i.e.,

   $$\operatorname{span}\left\{\begin{bmatrix}1\\2\\-3\end{bmatrix},\begin{bmatrix}0\\1\\-2\end{bmatrix}\right\}.$$

   (b) The reduced row echelon form of B is $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\end{bmatrix}$. The nullspace matrix of B is A, and thus $A=\begin{bmatrix}-1\\2\\1\end{bmatrix}$.

   (c) The projection matrix onto V is computed by

   $$P=\frac{1}{6}\begin{bmatrix}-1\\2\\1\end{bmatrix}\begin{bmatrix}-1 & 2 & 1\end{bmatrix}=\frac{1}{6}\begin{bmatrix}1 & -2 & -1\\ -2 & 4 & 2\\ -1 & 2 & 1\end{bmatrix}.$$

   (d) Because rank(P)=1, P has eigenvalues 0 and 1. P is semi-positive definite.

   (e) From (c), we have $(I-P)\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}-\frac{1}{6}\begin{bmatrix}1\\-2\\-1\end{bmatrix}=\frac{1}{6}\begin{bmatrix}5\\2\\1\end{bmatrix}$.

2.

   (a) rank(A^TA)=rank(A)=3 because all eigenvalues of A are nonzero.
   (b) det(2A)=2^3det $A=-48$ because $\det A=(-3)(1)(2)=-6$.
   (c) trace(2A)=2trace(A)=0 because $\operatorname{trace}(A)=(-3)+1+2=0$.
   (d) 9, 1, 4 because the eigenvalues of A^2 are the squares of eigenvalues of A.
   (e) NO, because the absolute values of some eigenvalues are greater than 1.
   (f) A-I has eigenvalues -4, 0, 1. Thus, (A-I) is not invertible.
   (g) NO. A and B do not have the same set of eigenvalues.
   (h) NO. Not necessarily so. For example, $A=\begin{bmatrix}-3 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$.

3.

(a) Because $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} [\mathbf{v}_1]_w$, $[\mathbf{v}_1]_w = \frac{1}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

(b) Similar to (a), we have $[\mathbf{v}_2]_w = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then,

$[\mathbf{v}_1]_w = \frac{1}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = M[\mathbf{v}_1]_v = M\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $[\mathbf{v}_2]_w = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = M[\mathbf{v}_2]_v = M\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, $M = \frac{1}{2}\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$.

(c) $A = MBM^{-1}$

(d) YES, because A and B are similar, as shown in (c).

4.

(a) $e^A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-3} & 0 \\ 0 & e^{-1} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2e^{-3} & 0 \\ e^{-3}-e^{-1} & 2e^{-1} \end{bmatrix}$

(b) When $c = 1$, A has repeated eigenvalues -2, and there is only one independent eigenvectors.

(c) Using the results in (a), we can solve the differential equation as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2e^{-3t} \\ e^{-3t}+e^{-t} \end{bmatrix}$$

(d) To ensure that the dynamic system is stable, we require that the real part of every eigenvalue is negative. The characteristic polynomial is

$p(\lambda) = \lambda^2 + 4\lambda + 3 + c$, which has roots $\lambda = -2 \pm \sqrt{1-c}$. Consider the following cases. (1) If $c<1$, there are two real eigenvalues, $\lambda = -2 \pm \sqrt{1-c}$. Stable system requires $c > -3$. (2) If $c \geq 1$, the real part of each eigenvalue is -2. Hence, the condition is simply $c > -3$.

5.

(a) $A = Q\Lambda Q^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

(b) When $\mathbf{x} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 2$.

(c) When $\mathbf{x} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 1$.