Linear Algebra
Solutions to Quiz 1 2005

1.

(a) A is a 4×3 matrix. Let $A=[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$. The complete solution indicates that

$$A\begin{bmatrix}3\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\quad A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\quad A\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

From the first equation, we can obtain the first column of A, $\mathbf{a}_1=\frac{1}{3}\begin{bmatrix}1\\2\\0\\1\end{bmatrix}$.

From the second equation, we know that $\mathbf{a}_1+\mathbf{a}_2=\mathbf{0}$, and thus $\mathbf{a}_2=-\mathbf{a}_1$. The third equation requires $2\mathbf{a}_1+\mathbf{a}_3=\mathbf{0}$, i.e., $\mathbf{a}_3=-2\mathbf{a}_1$. Hence,

$$A=\frac{1}{3}\begin{bmatrix}1&-1&-2\\2&-2&-4\\0&0&0\\1&-1&-2\end{bmatrix}.$$

(b) You can obtain the reduced row echelon form of A by applying Gaussian elimination. The reduced row echelon form of A is $R=\begin{bmatrix}1&-1&-2\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}$.

(c) There is only one nonzero row in R, which supplies a basis for the row space of A: $\left\{\begin{bmatrix}1\\-1\\-2\end{bmatrix}\right\}$.

(d) rank(A)=1 because there is only one vector in the basis for the row space of A.

2.

(a) TRUE. The reasoning is as follows:

$A\mathbf{x}=\mathbf{b}$ has exactly one solution for some $\mathbf{b}$

$\rightarrow \quad A\mathbf{x} = \mathbf{0}$ has trivial solution $\mathbf{x} = \mathbf{0}$ only.

$\rightarrow \quad N(A) = \{\mathbf{0}\} \rightarrow \quad \dim N(A) = 0$

$\rightarrow \quad \dim C(A) = \dim C(A^T) = n - \dim N(A) = n - 0 = n$

(b) $A\mathbf{x} = \mathbf{b}$ either has no solution or infinitely many solutions

$\rightarrow \quad C(A) \neq \mathbf{R}^m$ and $N(A) \neq \{\mathbf{0}\}$

$\rightarrow \quad \dim C(A) < m$ and $\dim N(A) > 0$

$\rightarrow \quad r = \dim C(A) < m$ and $r = \dim C(A^T) = n - \dim N(A) < n$

(c) FALSE. Think of $A = I$ and $B = 2I$. A and B have the identical four subspaces.

(d) FALSE. Suppose A is in its reduced row echelon form already,

rref(A) = A, say $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. But,

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \text{rref}(A)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}.$$

3.

(a) Represent A in the matrix multiplication as follows:

$$A = BC = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & a \end{bmatrix}.$$

It is clear that B is invertible. Imagine that B is the product of a sequence of elementary matrices, and thus $\text{rank}(A) = \text{rank}(C)$. The reduced row echelon form of C is $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and thus rank($A$)=2, regardless of a.

It is impossible to find a such that rank(A)=3.

(b) For the same reason as given in problem (a), we have N(A)=N(C). The nullspace matrix of A is thus $\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and the basis for $N(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(c) The first three columns of A are spanned by $1\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$.

The fourth column of A must be spanned by $1\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + a\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = k\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$.

Solving this equation yields k=1 and $a = -1$.