Linear Algebra
Solutions to Quiz 1 2007

1.

(a) To find the second column of A^{-1}, you need to solve the equation:

$$A\mathbf{x} = \begin{bmatrix} a & 1 & 0 \\ 1 & 2 & 1 \\ 2 & b & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{b},$$

where $\mathbf{x}$ is the second column of A^{-1}. Because the third column of A equals $\mathbf{b}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Reduce A to $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1-2a & -a \\ 0 & 0 & -2-a\dfrac{4-b}{1-2a} \end{bmatrix}$. The rank of A is 2 if $-2-a\dfrac{4-b}{1-2a}=0$, or $ab=2$. Otherwise, rank(A)=3.

(c) You are ask to solve for B so that

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} = C.$$

Usually, we can get B by computing $B = A^{-1}C$. But, if you examine the equation carefully, you will notice that B is simply a permutation matrix given by $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2.

(a) Since A is 4 by 3, the reasoning is as follows:

$$\dim C(A) = 3$$
$$\rightarrow \dim C(A^T) = 3$$
$$\rightarrow \dim N(A) = 3-3 = 0$$
$$\rightarrow N(A) = \{\mathbf{0}\}$$

(b) $\dim N(A^T) = 4 - 3 = 1 \rightarrow N(A^T) \neq \{\mathbf{0}\}$

(c) FALSE. For some **b**, A**x**=**b** has no solution. This is because $\dim C(A) = 3 < 4$. Therefore, $C(A) \neq \mathbf{R}^4$.

(d) FALSE. A has independent columns, but A has dependent rows. This is because four row vectors in $\mathbf{R}^3$ must be dependent.

3.

(a) If A is invertible, then A^T is also invertible. Hence, AA^T is invertible because $(AA^T)(A^T)^{-1}A^{-1} = I$.

(b) Suppose A is invertible. Then, $A^2A^{-1}A^{-1} = I$. However, A^2=0, and thus $A^2A^{-1}A^{-1} = 0$. This is a contradiction.

4.

(a) $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(b) Since $\dim C(A) = \text{rank}(A) = 3$, $\dim N(A^T) = 3 - \text{rank}(A) = 0$. Thus, $N(A^T) = \{\mathbf{0}\}$, and the basis is simply the empty set.

(c) $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(d) rank([A A])=3