Linear Algebra
Solutions to Quiz 1 2008

1.

(a) Apply Gaussian elimination to the augmented matrix as follows:

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & 4 & b_2 \\ 2 & 1 & 0 & -1 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & -3b_1 + b_2 + b_3 \end{bmatrix}$$

There are two pivots in the reduced row echelon form of A and thus rank(A)=2.

(b) A basis for the column space of A is given by its columns corresponding to the pivot columns of the reduced row echelon form: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(c) The simplest way you know thus far is to get the reduced row echelon form of A^T, which is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The nullspace matrix of A^T is $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$, and a basis for $N(A^T)$ is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(d) For those **b** in the column space of A, it requires $-3b_1 + b_2 + b_3 = 0$. The condition $C(A)=N(B)$ requires that $B = \begin{bmatrix} -3 & 1 & 1 \end{bmatrix}$.

2.

(a) The nullspace matrix of A is $N = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The reduced row echelon form of A can be obtained from N as $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) rank($\begin{bmatrix} A | \mathbf{b} \end{bmatrix}$)=2

(c) NO. A^TA is 4 by 4, but rank(A^TA)=rank(A)=2.

(d) NO. It is impossible to express $\begin{bmatrix} 2 \\ 3 \\ 6 \\ 1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(e) YES. Let $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and then B=AP. The equation $B\mathbf{x} = \mathbf{b}$ is equivalent to $AP\mathbf{x} = \mathbf{b}$. Let $\mathbf{y} = P\mathbf{x}$, and $\mathbf{x} = P^{-1}\mathbf{y} = P^T\mathbf{y}$. Since we already know the solution of $A\mathbf{y} = \mathbf{b}$, $\mathbf{x}$ is given by $\mathbf{x} = \begin{bmatrix} 6 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}$.

3.

(a) TRUE. A has full column rank and rank(A)=3. The left inverse C can be chosen as $C = (A^T A)^{-1} A^T$.

(b) FALSE. For example, A=I and B=$2I$. It is clear that $C(A)$=$C(B)$ and $N(A)$=$N(B)$.

(c) FALSE. If A and B have the same reduced row echelon form, then they have the identical nullspaces, $N(A)$=$N(B)$, but they may not have the identical left nullspaces. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.