Linear Algebra **Quiz 2**

5/16/2008

1. (40%) Let $A=\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. (Examine the matrix carefully before you answer the following questions.)

   (a) (10%) Find a basis for the orthogonal complement $C(A)^{\perp}$, where $C(A)$ denotes the column space of A.

   (b) (6%) Compute the rank of A^TA. (Please don't do a lot of calculations.)

   (c) (8%) What is the closest vector in the orthogonal complement $C(A)^{\perp}$ to $\mathbf{b}=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$?

   (d) (8%) What is the closest vector in the column space $C(A)$ to $\mathbf{b}$?

   (e) (8%) Let P be the 3 by 3 projection matrix onto $C(A)$, and P' be the 3 by 3 projection matrix onto $C(A)^{\perp}$. Is $PP'=0$? Yes or no. Briefly explain why.

2. (30%) Answer the following questions.

   (a) Let $A=\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$, $B=\begin{bmatrix} 1+a+a^2 & 1+a & 1 \\ 1+b+b^2 & 1+b & 1 \\ 1+c+c^2 & 1+c & 1 \end{bmatrix}$, $C=k\begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{bmatrix}$.

   (7%) If det A=5, what is the determinant of B?

   (7%) If det A=5 and det C=-40, what is the value of k?

(b) Let $C = \begin{bmatrix} c & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ be the cofactor matrix of A.

(8%) Suppose that (1, 1) entry of A^{-1} is 2, i.e., $A^{-1} = \begin{bmatrix} 2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$. If c=4, find the determinant of A.

(8%) Find c such that $x_1 = 1$ for the solution of $C\mathbf{x} = \begin{bmatrix} c & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

3. (30%) . Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation satisfying

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The v-basis is given by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The w-basis is given by $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) (8%) Find the matrix M that maps the coordinates in v-basis to the coordinates in w-basis, i.e., $[\mathbf{x}]_w = M[\mathbf{x}]_v$.

(b) (8%) What is the matrix A expressing T in terms of the v-basis? Note that $\left[T(\mathbf{x})\right]_v = A[\mathbf{x}]_v$.

(c) (8%) What is the matrix B expressing T in terms of the w-basis? Note that $\left[T(\mathbf{x})\right]_w = B[\mathbf{x}]_w$.

(d) (6%) Describe the relationship between A and B with one equation.