Linear Algebra
Solutions to Quiz 2 2007

1.

(a) The row space of A is spanned by $\mathbf{u}=\begin{bmatrix}1\\3\\1\end{bmatrix}$. The projection matrix P onto the line along $\mathbf{u}$ is thus $P=\frac{\mathbf{u}\mathbf{u}^T}{\|\mathbf{u}\|^2}=\frac{1}{11}\begin{bmatrix}1&3&1\\3&9&3\\1&3&1\end{bmatrix}$.

(b) $\mathbf{p}=P\mathbf{b}=\frac{1}{11}\begin{bmatrix}1&3&1\\3&9&3\\1&3&1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\frac{3}{11}\begin{bmatrix}1\\3\\1\end{bmatrix}$

(c) Since any vector on the row space of A will project onto itself, we have $PA^T=A^T$. Because P is symmetric, $P^T=P$, we must have $A=(PA^T)^T=AP^T=AP$. Also, you may get the required relationship from the formula of P, $P=A^T(AA^T)^{-1}A$. Direct computation shows that $AP=AA^T(AA^T)^{-1}A=A$.

(d) The nullspace is the orthogonal complement to the row space. The reduced row echelon form of A is $\begin{bmatrix}1&3&1\\0&0&0\end{bmatrix}$. The nullspace matrix is $\begin{bmatrix}-3&-1\\1&0\\0&1\end{bmatrix}$. Use Gram-Schmidt process to find an orthonormal basis for $N(A)$. Let $\mathbf{u}_1=\mathbf{n}_1=\begin{bmatrix}-3\\1\\0\end{bmatrix}$.

Then, $\mathbf{u}_2=\mathbf{n}_2-\frac{\mathbf{n}_2^T\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1=\begin{bmatrix}-1\\0\\1\end{bmatrix}-\frac{3}{10}\begin{bmatrix}-3\\1\\0\end{bmatrix}=\frac{1}{10}\begin{bmatrix}-1\\-3\\10\end{bmatrix}$. Therefore, an orthonormal basis for $N(A)$ is $\left\{\frac{1}{\sqrt{10}}\begin{bmatrix}-3\\1\\0\end{bmatrix},\frac{1}{\sqrt{110}}\begin{bmatrix}-1\\-3\\10\end{bmatrix}\right\}$.

2.

(a) rank(Q)=n since Q has n independent columns.

(b) Since $C(P)$=$C(Q)$, we have dim $C(P)$=dim $C(Q)$=rank(Q)=n.

(c) TRUE. By the relations rank(A^TA)=rank(A) and rank(A^T)=rank(A), we have rank(Q^TQ)=rank(Q) and rank(QQ^T)=rank(Q^T), and thus rank(Q^TQ)=rank(QQ^T).

3.

(a) From $x_3 = \dfrac{\begin{vmatrix} 1 & 1 & b_1 \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{vmatrix}}{\det A} = 0,$ we have $b_1 = -2.$

(b) Recall that $A^{-1} = \dfrac{C^T}{\det A}.$ Then, $\det C = (\det A)^{n-1} = (-2)^2 = 4.$

(c) Given $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$ and $B = [\mathbf{u}-\mathbf{v} \quad \mathbf{u}+\mathbf{v} \quad \mathbf{u}+2\mathbf{v}-\mathbf{w}]$, B can be expressed as

follows: $B = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = AC.$ Because A is invertible (A has

independent columns) and C is invertible (det C=-2), B is also invertible. So, B has independent columns.

(d)

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$