Linear Algebra **Quiz 2**

5/16/2008

1.

(a) Note that $C(A)^{\perp} = N(A^T)$. The reduced row echelon form of A^T is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, and

thus the nullspace matrix of A^T is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, which forms a basis for $C(A)^{\perp}$.

(b) A only has two independent columns, and thus rank(A)=2. Thus, rank(A^TA)=rank(A)=2.

(c) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. The projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto $C(A)^{\perp}$ is $\mathbf{p} = \dfrac{\mathbf{u}^T\mathbf{b}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} = \dfrac{1}{2}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

(d) From (c), the projection of **b** onto the column space of A is simply

$$\mathbf{b} - \mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

(e) YES. Note that $P' = I - P$. Then, $PP' = P(I-P) = P - P^2 = P - P = 0$.

2.

(a)

(1) $$\det B = \begin{vmatrix} 1+a+a^2 & 1+a & 1 \\ 1+b+b^2 & 1+b & 1 \\ 1+c+c^2 & 1+c & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -\det A = -5$$

(2) Since $\det C = k^3 \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = k^3 \det A = -40$, we get $k = -2$.

(b)

(1) Because $A^{-1} = \dfrac{C^T}{\det A}$, we can conclude that 2=4/det A, and thus $\det A = 2$.

(2) By Cramer's rule, $x_1 = \dfrac{\begin{vmatrix} 3 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}}{\det C} = \dfrac{3}{\det C} = 1$, implying that det C=3=c(1)(1), and thus c=3.

3.

(a) M can be solved by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, which yields $M = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) Because $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, we have

$$A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}.$$

(c) $B = MAM^{-1} = \begin{bmatrix} 0 & 1 \\ 2 & 7 \end{bmatrix}$

(d) $BM = MA$