Linear Algebra **Final Exam**

6/22/2009

1. (17%) Let L be the line in $\mathbf{R}^3$ spanned by the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Let P be the projection matrix for the projection onto the line L, and Q be the projection matrix projecting onto the orthogonal complement $L^{\perp}$ to the line L.
   (a) (3%) What are the three eigenvalues of the matrix P?
   (b) (3%) What are the three eigenvalues of the matrix Q?
   (c) (3%) Is the matrix P diagonalizable? Yes or no. State your reason briefly.
   (d) (4%) Find an orthonormal basis of the orthogonal complement $L^{\perp}$ to the line L.
   (e) (4%) What is the closest vector in the orthogonal complement $L^{\perp}$ to $\begin{bmatrix}1\\2\\3\end{bmatrix}$?

2. (30%) Suppose A is a 3 by 3 matrix having eigenvalues $0, -1, -2,$ and the corresponding eigenvectors are **u**, **v**, **w**, respectively. Answer the following questions.
   (a) (3%) What is the dimension of the nullspace of A?
   (b) (3%) What is the determinant of $A^2 - I$? (Hint: Factorize it!)
   (c) (3%) What is trace of A^2?
   (d) (3%) What are the eigenvalues of $(I - A)^{-1}$?
   (e) (3%) What is the rank of the eigenvector matrix S? The three columns of S are the eigenvectors **u**, **v**, and **w**.
   (f) (3%) Find a basis for the column space of A.
   (g) (4%) Show that the eigenvalues of e^A are e^0, e^{-1}, e^{-2}.

   Yes or no. State your reason briefly or give a counterexample.
   (h) (4%) Suppose A is symmetric. If B similar to A, is B also symmetric?
   (i) (4%) Can A be an orthogonal matrix?

3. (12%) Suppose Q is an m by n matrix with $Q^TQ=I$.

(a) (4%) What is the rank of Q?

(b) (4%) Write down the projection matrix onto the column space of Q.

(c) (4%) If $\det\left(QQ^T\right)=0$, what does this tell you about the relationship between m and n?

4. (15%) Consider the 3 by 3 matrix $A=\begin{bmatrix} c & 2 & 1 \\ 2 & c & 1 \\ 1 & 1 & 2 \end{bmatrix}$, where c is a real number.

(a) (5%) For which values of c is the matrix A singular?

(b) (5%) For which values of c is the matrix A positive definite?

(c) (5%) For which values of c is the exponentials e^{At} converging to the zero matrix as $t \to \infty$? (Hint: A is negative definite iff $-A$ is positive definite.)

5. (10%) Suppose A is a 3 by 3 matrix having real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and independent eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$.

(a) (4%) Write down the general form of the solutions to $\mathbf{u}_{k+1} = A\mathbf{u}_k$, in terms of the eigenvalues and eigenvectors.

(b) (3%) Suppose every solution to $\mathbf{u}_{k+1} = A\mathbf{u}_k$ approaches $c\mathbf{x}_1$ as $k \to \infty$, where c depends on $\mathbf{u}_0$. What does this tell you about $\lambda_1, \lambda_2, \lambda_3$?

(c) (3%) Suppose every solution to $\mathbf{u}_{k+1} = e^{A}\mathbf{u}_k$ approaches the zero vector as $k \to \infty$. What does this tell you about $\lambda_1, \lambda_2, \lambda_3$?

6. (16%) Consider the 2 by 2 matrix

$$A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(a) (6%) Find the eigenvalues and corresponding eigenvectors of A.

(b) (5%) Find the unit vector $\mathbf{x}$ ($\|\mathbf{x}\|=1$) at which $\|A\mathbf{x}\|$ is maximized. What is the maximum value of $\|A\mathbf{x}\|$? (Two questions.)

(c) (5%) List every class of matrices to which the matrix A belongs: diagonalizable, invertible, symmetric, orthogonal, projection.