

1. (36%) Suppose the block matrix  $[A \ I_3]$  is reduced by elementary row operations to

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- (a) (6%) Is  $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  solvable? Why or why not.
- (b) (6%) What is the first column of  $A$ ?
- (c) (8%) Can the third and fourth columns of  $A$  form a basis for the column space of  $A$ ? Explain briefly.
- (d) (8%) Find a matrix  $B$  such that the column space of  $B$  equals the nullspace of  $A$ , i.e.,  $C(B)=N(A)$ .
- (e) (8%) Find all the solutions of  $A^T\mathbf{x} = \mathbf{0}$ .

2. (34%) Suppose  $A$  is a 4 by 3 matrix of the form  $A = \begin{bmatrix} ? & 1 & ? \\ ? & 1 & ? \\ ? & 1 & ? \\ ? & 1 & ? \end{bmatrix}$ . The general

solution to the equation  $A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  is  $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

- (a) (10%) Determine  $A$ .
- (b) (8%) Find a basis for the row space of  $A$ .
- (c) (8%) If the solution to the above equation is in the row space of  $A$ , what is it?
- (d) (8%) What is the rank of the block matrix  $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$ ?
3. (30%) True or false. Give a reason or a counterexample.  
Suppose  $A$  and  $B$  are 5 by 3 matrices.
- (a) (10%) If  $A$  and  $B$  have the same reduced row echelon form  $R$ , then they have the identical column space.
- (b) (10%) It is impossible to find  $A$  and  $B$  such that  $AB^T = I_5$ .
- (c) (10%) If  $A^T A = 0$ , then  $A = 0$ . (Hint: what is the rank of  $A^T A$ ?)