Linear Algebra
Solutions to Quiz 1 2009

1.

(a) NO. Let $\mathbf{b} = (0, 0, 1)^T$. The augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is reduced to

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$ To have solutions, the last row must be a zero row.

Hence, $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not solvable.

(b) The first column of A is $(0, 0, 0)^T$. Let the given reduced echelon block matrix be $\begin{bmatrix} R & E \end{bmatrix}$. Then, $EA = R$, or $A = E^{-1}R$. The first column of A can be obtained by multiplying E^{-1} by the first column of R, which is a zero column.

(c) YES. Let the columns of A be $\mathbf{a}_i$. From R, we know that the second and third columns of A are linearly independent. Also, $\mathbf{a}_4 = 2\mathbf{a}_2 + \mathbf{a}_3$. Since $\mathbf{a}_2$ and $\mathbf{a}_3$ are independent, it follows that $c_1\mathbf{a}_3 + c_2\mathbf{a}_4 = c_1\mathbf{a}_3 + c_2(2\mathbf{a}_2 + \mathbf{a}_3) = (2c_2)\mathbf{a}_2 + (c_1 + c_2)\mathbf{a}_3 = \mathbf{0}$ only has trivial solutions. That is, $2c_2 = 0$ and $c_1+c_2 = 0$, implying $c_1 = c_2 = 0$.

(d) The nullspace matrix N satisfies $RN = 0$, implying that the columns of N spans the nullspace of A (or R). From R, we obtain $B = N = \begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$.

(e) The equivalent problem is to solve $\mathbf{x}^T A = \mathbf{0}^T$. With the relationship $EA = R$, the rows of E generating zero rows in R form the basis of the solution space. The solutions are all linear combinations of $(0, 0, 1)^T$.

2.

(a) From $A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, we know that the first column of A is $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. From

$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, we know that the sum of all three columns of A is zero, and

thus the third column is $\begin{bmatrix}-2\\-2\\-1\\-1\end{bmatrix}$. Therefore, $A=\begin{bmatrix}1&1&-2\\1&1&-2\\0&1&-1\\0&1&-1\end{bmatrix}$.

(b) From (a), A can be reduced to $R=\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\\0&0&0\end{bmatrix}$. Note that the reduced row echelon form can also be obtained from the nullspace matrix $N=\begin{bmatrix}1\\1\\1\end{bmatrix}$. The two nonzero (pivot) rows give a basis for the row space of A:

$\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$.

(c) The particular solution in the row space of A requires that the general solution $\mathbf{x}=\begin{bmatrix}1\\0\\0\end{bmatrix}+\alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$ be expressed as a linear combination of the two basis vectors of $C(A^T)$. That is, $\begin{bmatrix}1\\0\\0\end{bmatrix}+\alpha\begin{bmatrix}1\\1\\1\end{bmatrix}=c\begin{bmatrix}1\\0\\-1\end{bmatrix}+d\begin{bmatrix}0\\1\\-1\end{bmatrix}$, giving the following equations: $1+\alpha=c,\ \alpha=d,\ \alpha=-c-d$. Solving them yields $\alpha=-\frac{1}{3},\ c=\frac{2}{3},\ d=-\frac{1}{3}$. Hence, $\mathbf{x}_p=\begin{bmatrix}1\\0\\0\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2\\-1\\-1\end{bmatrix}$.

(d) Appling row operations to $\begin{bmatrix}A&A\\A&A\end{bmatrix}$, we can get $\begin{bmatrix}R&R\\0&0\end{bmatrix}$. The number of nonzero rows of the reduced block matrix equals the number of nonzero rows of R, which is 2. Therefore, $\operatorname{rank}\begin{bmatrix} A & A \\ A & A \end{bmatrix} = 2$.

3.

(a) FALSE. The reduced row echelon form R indicates the interrelationship between the columns of A, but it will destroy the column space of A. For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) TRUE. Think of the problem as solving the following linear equation:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{b}$$

where **b** is a 5-dimsional unit vector with four zero components. Since A only has three columns, the column space cannot fill the entire $\mathbf{R}^5$ space. The above linear equation is not solvable for some **b**'s.

(c) TRUE. Note that rank(A^TA) = rank(0)=0. Since rank(A^TA) = rank(A), it follows that rank(A) = 0, implying that the column space of A only contains the zero vector. Therefore, A = 0.