

1. (40%) Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}$. Suppose P is the 4 by 4 projection matrix onto the row space of A .

- (a) (8%) What is the closest vector in the column space $C(A)$ to $\mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$?
- (b) (10%) Find an orthonormal basis for the row space of A .
- (c) (6%) What is the rank of P ?
- (d) (8%) Express the projection matrix P in terms of A . (What does the projection matrix P do?)
- (e) (8%) What is the projection matrix onto the nullspace of $A^T A$? (Actually, you don't need to compute $A^T A$.)

2. (30%) Answer the following questions.

- (a) (7%) If $\det A > 0$, find all possible values of a .

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & a & 0 \\ 1 & 1 & a & 0 & 0 \\ 1 & a & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \end{bmatrix}$$

- (b) (7%) Find the determinant of B :

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- (c) (8%) Suppose Q is a 5 by 3 matrix with orthonormal columns. Is QQ^T a projection matrix? Yes or no. (Check the two properties of projections.)
- (d) (8%) True or false: Let A be an n by n real matrix, and C be the cofactor matrix of A . If A is invertible and $C = A$, then A is an orthogonal matrix. (Note that the cofactor matrix C is the transpose of the adjoint of A .) State your reasoning.

3. (30%) . Answer the following questions.

(a) (8%) Suppose T is a linear transformation from \mathbf{P}^2 to \mathbf{P}^3 , described by

$$T(f(x)) = f(x) - xf'(x).$$

Find a basis for the range of T . Represent the basis as a set of coordinate vectors with respect to the standard basis $\{1, x, x^2, x^3\}$.

(b) (8%) Suppose L is a linear transformation and $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Express the matrix representation of L with respect to the standard basis.

(c) (7%) Suppose Q is an n by n matrix and $Q = I - 2\mathbf{u}\mathbf{u}^T$, with $\mathbf{u}^T\mathbf{u} = 1$. What is Q^{99} ?

(d) (7%) True or false: Suppose $T : \mathbf{R}^4 \rightarrow \mathbf{R}^3$ is a linear transformation. If T maps \mathbf{R}^4 onto \mathbf{R}^3 , then T is one-to-one. If yes, state your reason; otherwise, give a counterexample.