Linear Algebra **Quiz 2**

5/13/2009

1.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, indicating that A has three linearly independent columns, i.e., $C(A) = R^3$. The closest vector in the column space to $\mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$ is simply **b** itself.

(b) From the reduced row echelon form obtained in (a), the row space of A is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Use the Gram-Schmidt process to find an orthonormal basis for the row space. Let $\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1^T \mathbf{v}_1}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Note that $\mathbf{v}_3$ is already orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus, $\mathbf{u}_3 = \mathbf{v}_3$. The three orthonormal basis vectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) Because $C(P) = C(A^T)$, we have rank(P) = rank(A^T) = rank(A). From (a), it is seen that A has three pivots, and thus the rank of P is 3.

(d) Because P is the projection matrix onto $C(A^T)$, we can think of the least-squares approximation to $B\mathbf{x} = \mathbf{b}$, where $B = A^T$. The normal equation is then $B^T B\hat{\mathbf{x}} = B^T\mathbf{b}$. Solving yields $\hat{\mathbf{x}} = (B^T B)^{-1} B^T \mathbf{b}$, and thus $B\hat{\mathbf{x}} = B(B^T B)^{-1} B^T \mathbf{b} = P\mathbf{b}$. Therefore, the projection matrix is $P = A^T (AA^T)^{-1} A$.

(e) Since the nullspace of A^TA equals the nullspace of A, we can compute the projection matrix onto $N(A)$. From the reduced row echelon form in (a), the nullspace matrix is

$N = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$. The nullspace of A is spanned by the vector $\mathbf{n} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, so the projection

matrix is $\dfrac{\mathbf{n}\mathbf{n}^T}{\mathbf{n}^T\mathbf{n}} = \dfrac{1}{6}\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$.

2.

(a) Exchange the rows of A to get a triangular form as follows:

$$\det A = -\begin{vmatrix} a & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & a & 0 \\ 1 & 1 & a & 0 & 0 \\ 1 & a & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & a \end{vmatrix} = \begin{vmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 \\ 1 & 1 & a & 0 & 0 \\ 1 & 1 & 1 & a & 0 \\ 1 & 1 & 1 & 1 & a \end{vmatrix} = a^5$$

To have positive det A, it is required that $a > 0$.

(b) Let $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The column space of D is spanned by only two vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This means that D is singular.

Hence, det B = det (CDE) = (det C)(det D)(det E) = 0.

(c) Any projection matrix P satisfies (1) $P^T = P$ and (2) $P^2 = P$. QQ^T is a projection matrix because $(QQ^T)^T = QQ^T$ and $(QQ^T)^2 = QQ^TQQ^T = QI_3Q^T = QQ^T$.

(d) FALSE. Note that $AC^T = (\det A)\, I$. If $C = A$, then $AA^T = (\det A)I$. Taking determinant on both sides, $\det (AA^T) = (\det A)(\det A^T) = (\det A)^2$ and $\det(AA^T) = (\det A)^n$. Hence, $(\det A)^{n-2} = 1$. If n is odd, then det A = 1, and thus A is an orthogonal matrix because $A^T = A^{-1}$; if n is even and $n > 2$, then det A = -1 or 1, and in this case, A is not an orthogonal matrix when det A = -1.

3.

(a) Compute $T(1) = 1$, $T(x) = 0$, $T(x^2) = -x^2$. The range of T is spanned by the coordinate vectors: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\0\end{bmatrix}$.

(b) Note that

$$L\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\frac{1}{2}L\left(\begin{bmatrix}1\\1\end{bmatrix}\right)+\frac{1}{2}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1/2\\1/2\\1\end{bmatrix} \text{ and}$$

$$L\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\frac{1}{2}L\left(\begin{bmatrix}1\\1\end{bmatrix}\right)-\frac{1}{2}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}.$$

Then, the matrix representation is $\left[L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \quad L\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]=\begin{bmatrix}1/2 & 1/2\\1/2 & -1/2\\1 & 0\end{bmatrix}$.

(c) Note that Q is a reflection matrix. Because $\mathbf{u}^T\mathbf{u}=1$, we obtain $Q^2=(I-2\mathbf{u}\mathbf{u}^T)(I-2\mathbf{u}\mathbf{u}^T)=I-4\mathbf{u}\mathbf{u}^T+4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T=I$. Therefore, $Q^{99}=Q^{98}\,Q=Q$.

(d) FALSE. For example, represent T as in matrix form as $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$. Clearly, the range of T is $\mathbf{R}^3$, but the kernel of T is spanned by $(0, 0, 0, 1)^T$. Hence, T is not one-to-one.