**Linear Algebra**
**Solutions to Exam 1 2010**

1.

(a) Since $L$ is invertible, $A$ is row equivalent to $U$. Use row operations to reduce $U$. If $k=0$, we have $U \to \begin{bmatrix} 1 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and rank($A$)=1. If $k=1$, we have $U \to \begin{bmatrix} 1 & 1/2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and rank($A$)=2, If $k \neq 0,1$, $U \to \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and rank($A$)=3.

(b) When $k=1$, rank($A$) = 2. In this case, the reduced row echelon form becomes $R = \begin{bmatrix} 1 & 1/2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. From $R$, we know that the first and third columns of $A$ form a basis for $C(A)$. Compute

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 2 \\ -2 & -1 & 1 & -1 \\ 4 & 2 & -1 & 3 \end{bmatrix}.$$

Hence, the basis is

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(c) YES. Consider $A\mathbf{x} = LU\mathbf{x} = \mathbf{0}$. Multiplying $L^{-1}$, we have $U\mathbf{x} = \mathbf{0}$.

2.

(a) It is impossible to tell whether $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is solvable or not. From the reduced row echelon form, we can only tell that the first and third columns

of $A$ are independent, but the column space of $A$ cannot be determined by the reduced row echelon form.

(b) The column space of $A^T$ is the row space of $A$. The nonzero rows of the reduced row echelon form of $A$ provide a basis for $C(A^T)$: $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Clearly, $\begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \end{bmatrix}$ is in $C(A^T)$, so the equation is solvable.

(c) The particular solution is $\mathbf{x}_p = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. From the given reduced row echelon form, we know the nullspace matrix is $N = \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$, and thus the homogeneous solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x}_h = \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. Therefore, the general solution is $\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

3.

(a) TRUE. If $A$ and $B$ have the same reduced row echelon form, then they have the same number of pivots, implying that they have the same rank.

(b) TRUE. Taking transpose of $AB^T = 0$ yields $BA^T = 0^T$. The columns of $A^T$, i.e., the rows of $A$, are in the nullspace of $B$.

(c) FALSE. The nullspace is not determined by the column space. For example, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. The nullspace of $A$ is spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ while the nullspace of } B \text{ is spanned by } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(d) TRUE. Because rank($A$) = rank($A^T A$), we have rank($A$) = rank($I_3$)=3. Since $A$ is 4 by 3 and the dimension of the column space is 3, the columns of $A$ must be linearly independent.