Linear Algebra **Exam 2**

5/17/2010

1. (36%) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & a \\ -2 & 1 \end{bmatrix}$. It is known that the orthogonal projection matrix onto the column space of A is $P = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

   (a) (8%) What is the rank of the orthogonal projection matrix onto the row space of A? State your reasoning briefly.

   (b) (8%) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto the nullspace of A^T?

   (c) (10%) Find a 3 by 2 matrix Q containing orthonormal columns so that $P = QQ^T$.

   (d) (10%) Find the least-squares solution of $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

2. (32%) Answer the following questions.

   (a) (8%)

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{vmatrix} = ?$$

   (b) (8%) If $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, what is P^{50}?

   (c) (8%) Let C be the cofactor matrix of P, where P is given in (b). What is $\det C$?

   (d) (8%) True or false: If A is an m by n matrix and $\det(A^T A) = 0$, then $\det(AA^T) = 0$. Give a reason or counterexample.

3. (32%) . Answer the following questions.

(a) (12%) Suppose T is a linear transformation from $\mathbf{P}^2$ to $\mathbf{R}^2$, and $T(1)=\begin{bmatrix}1\\1\end{bmatrix}$,

$T(1+x)=\begin{bmatrix}2\\2\end{bmatrix}$, $T(1-x+x^2)=\begin{bmatrix}2\\2\end{bmatrix}$. Find the kernel of T, i.e., the set of $\mathbf{p}$ in $\mathbf{P}^2$ such that $T(\mathbf{p})=\mathbf{0}$.

(b) (12%) If the matrix of a linear transformation T on $\mathbf{R}^3$, with respect to the standard basis is $\begin{bmatrix}0 & 1 & 1\\1 & 0 & -1\\-1 & -1 & 0\end{bmatrix}$, what is the matrix of T with respect to the basis

$$B=\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}?$$

(c) (8%) True or false: Suppose $T:\mathbf{R}^3\to\mathbf{R}^4$ is a linear transformation. If T is not one-to-one, then T maps onto $\mathbf{R}^4$. If true, state your reason; otherwise, give a counterexample.