Linear Algebra **Exam 2**

5/17/2010

1.

(a) The column space of the orthogonal projection matrix P equals to the column space of A. This is also true for the row space. Thus the rank of the orthogonal projection matrix onto the row space of A equals to the dimension of the row space, which is the rank of A. Since rank(A) = rank(P) = 2, the answer is 2.

(b) Note that the nullspace of A^T is the orthogonal complement of the column space of A.

The orthogonal projection matrix onto $N(A^T)$ is $I - P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, the

orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $N(A^T)$ is $(I-P)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(c) The matrix Q contains the orthonormal bases for the column space of the orthogonal projection matrix P. The first and third columns of P are orthogonal. Hence, one

possible choice is $Q = \begin{bmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$.

(d) Instead of finding the value of a first, project $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ onto the column space of A by

computing $P\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$. The solution of $A\hat{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ -1 & a \\ -2 & 1 \end{bmatrix}\hat{\mathbf{x}} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$ is the

least-squares solution. Solving it yields $\hat{\mathbf{x}} = \begin{bmatrix} 1/6 \\ 1/3 \end{bmatrix}$. Also, it is found that $a = -1$.

2.

(a) You can apply a sequence of elementary row operations to reduce the given matrix into triangular form:

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{vmatrix} = \cdots$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{vmatrix} = 6 \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = -6.$$

(b) Note that the permutation matrix P has the row sequence (2,4,1,3). Start with (1,2,3,4), the sequence of row permutation is as follows:

$$(1,2,3,4) \to (2,4,1,3) \to (4,3,2,1) \to (3,1,4,2) \to (1,2,3,4)$$

This indicates that $P^4 = I$, and thus $P^{50} = (P^4)^{12} P^2 = P^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(c) Recall that P and its cofactor matrix C are related by $P^{-1} = \dfrac{C^T}{\det P}$, or

$PC^T = (\det P) I_4$. Taking determinant,

$\det(PC^T) = (\det P)(\det C^T) = (\det P)(\det C) = (\det P)^4$.

Hence, $\det C = (\det P)^3 = (-1)^3 = -1$.

(d) FALSE. Note that $\operatorname{rank}(A^T A) = \operatorname{rank}(A) = \operatorname{rank}(A^T) = \operatorname{rank}(AA^T)$, $A^T A$ is n by n while AA^T is m by m. If $n \neq m$, it is possible that one is singular and the other is nonsingular. For example, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then $A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

3.

(a) Compute $T(x) = T(1+x) - T(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $T(x^2) = T(1-x+x^2) - T(1) + T(x) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. The matrix representation of T is $A = \begin{bmatrix} T(1) & T(x) & T(x^2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. The nullspace of A is spanned by $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, and therefore the kernel of T is spanned by $\{-1+x, -2+x^2\}$.

(b) Compute $T(\mathbf{x}_1) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \mathbf{x}_1$,

$T(\mathbf{x}_2) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$, $T(\mathbf{x}_3) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -\mathbf{x}_3$.

Thus the matrix of T with respect to the new basis is

$$\begin{bmatrix} [T(\mathbf{x}_1)]_B & [T(\mathbf{x}_2)]_B & [T(\mathbf{x}_3)]_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(c) FALSE. For example, T can never map onto $\mathbf{R}^4$, because if so, then the rank of T would be 4, which is large than the dimension of the domain $\mathbf{R}^3$.