Linear Algebra **Final Exam**

6/21/2010

You must show your work to get credit.

1. (22%) Let A be a 4 by 3 matrix. The general solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) (3%) Determine the rank of A.

   (b) (4%) Find an orthonormal basis for the nullspace of A.

   (c) (4%) What is the closest vector in the column space of A to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

   (d) (4%) Is $A^T\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ solvable? Yes or no. State your reason briefly.

   (e) (4%) Let Q be the orthogonal projection matrix onto the row space of A. Find all the eigenvectors of Q corresponding to eigenvalue 0.

   (f) (3%) Is Q described in (e) an orthogonal matrix? Yes or no. State your reason briefly.

2. (34%) Suppose A is a 3 by 3 matrix having eigenvalues $-3, -2, 1$ and the corresponding orthonormal eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

   respectively. Answer the following questions.

   (a) (4%) What is the determinant of $2A + I$?

   (b) (6%) Find bases for the column space and the nullspace of $A + 2I$, respectively. (Two questions.)

   (c) (5%) Find $(A^{-1})^k$, as $k \to \infty$.

   (d) (4%) Is it true that $\det(e^A) = e^{\text{trace}(A)}$? Yes or no. Give your reasoning.

   (e) (5%) Let $\mathbf{x} \in \mathbf{R}^3$ be a unit vector, i.e., $\|\mathbf{x}\| = 1$. What is the maximum value of $\|A\mathbf{x}\|$?

(f) (5%) Let $\mathbf{x} \in \mathbf{R}^3$ be a unit vector, i.e., $\|\mathbf{x}\| = 1$. What is the minimum value of $\mathbf{x}^T A\mathbf{x}$?

(g) (5%) List every class of matrices to which the matrix *A* belongs: nonsingular, symmetric, orthogonal, diagonalizable, positive definite.

3. (20%) True or false. State your reason briefly or give a counterexample. Let *A* be an *n* by *n* real matrix.

(a) (4%) For every positive integer *k*, A^k is similar to *A*.

(b) (4%) If *A* is symmetric, then e^A is positive definite.

(c) (4%) Suppose the LDU factorization of *A* is $A = LDU$. The diagonal entries of *D* are the eigenvalues of *A*.

(d) (4%) If *A* is symmetric and $\det A > 0$, then *A* is positive definite.

(e) (4%) If $A + A^T = 0$, then *A* is singular.

4. (12%) Let $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$. For each of the following questions, describe the condition of *a*, where *a* is a real number.

(a) (4%) *A* is positive definite.

(b) (4%) The differential equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ is always stable.

(c) (4%) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for every nonzero $\mathbf{x} \in \mathbf{R}^2$.

5. (12%) Consider a linear transformation *T* represented by the following 2 by 2 matrix with respect to the standard basis:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Suppose the *v*-basis is given by $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the *w*-basis is given by $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) (4%) What are the coordinate vectors of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with respect to *w*-basis? (Two questions.)

(b) (4%) Let *B* be the corresponding matrix for *T* with respect to *v*-basis. Find *B*.

(c) (4%) Let *C* be the corresponding matrix for *T* with respect to *w*-basis. Find the transformation matrix *D* such that $B = DCD^{-1}$.