Linear Algebra
Solutions to Final Exam 2010

1.

(a) The nullspace of A is spanned by $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\0\\1\end{bmatrix}$, and thus $\dim N(A)=2$.

By the rank-nullity theorem, $\operatorname{rank}(A)=n-\dim N(A)=3-2=1$.

(b) Let $\mathbf{x}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}, \mathbf{x}_2=\begin{bmatrix}3\\0\\1\end{bmatrix}$. Set $\mathbf{u}_1=\mathbf{x}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}$. Compute

$\mathbf{u}_2=\mathbf{x}_2-\frac{\mathbf{u}_1^T\mathbf{x}_2}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1=\begin{bmatrix}3\\0\\1\end{bmatrix}-\frac{-3}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3\\3\\2\end{bmatrix}$. An orthonormal basis for the

nullspace of A is given by $\mathbf{q}_1=\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \mathbf{q}_2=\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}=\frac{1}{\sqrt{22}}\begin{bmatrix}3\\3\\2\end{bmatrix}$.

(c) Since $\operatorname{rank}(A)=1$, the column space of A is spanned by $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$. The

closest vector in $C(A)$ to $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is its projection on $C(A)$, i.e.,

$$\frac{1}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}.$$

(d) Take the transpose of $A^T\mathbf{y}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and get $\mathbf{y}^TA=\begin{bmatrix}1 & 1 & 1\end{bmatrix}$. It is

equivalent to ask whether $\begin{bmatrix}1 & 1 & 1\end{bmatrix}$ is in the row space of A. Note that the row space of A is the orthogonal complement of the nullspace of A. But the inner product of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}3\\0\\1\end{bmatrix}$ is 4, so the equation is not solvable.

(e) Let $\mathbf{x}$ be the eigenvector of Q satisfying $Q\mathbf{x} = \mathbf{0}$. This indicates that $\mathbf{x}$ is orthogonal to the column space of Q, which is the row space of A. Hence, the eigenspace corresponding to eigenvalue 0 is the nullspace of A, which is spanned by $\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}3\\0\\1\end{bmatrix}$.

(f) NO. Since Q has zero eigenvalue, it is singular, and thus can't be an orthogonal matrix.

2.

(a) Note that the eigenvalues of $2A+I$ are $-5,-3,3,$ and thus $\det(2A+I)=(-5)(-3)(3)=45$.

(b) Note that the eigenvalues of $A+2I$ are $-1,0,3,$ and the corresponding eigenvectors are the same as those of A. Therefore, $\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$ form a basis for the column space of A and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ forms a basis for the nullspace of A.

(c) Let $Q=\begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3\end{bmatrix}$ denote the eigenvector matrix of A. Then, A can be orthogonally diagonalized as $A=Q\Lambda Q^T=Q\begin{bmatrix}-3 & & \\ & -2 & \\ & & 1\end{bmatrix}Q^T$. Then,

$$(A^{-1})^k = Q\begin{bmatrix} \left(-\frac{1}{3}\right)^k & & \\ & \left(-\frac{1}{2}\right)^k & \\ & & 1^k \end{bmatrix}Q^T \text{ approaches to}$$

$$Q\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}Q^T = \mathbf{q}_3\mathbf{q}_3^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ as } k \to \infty.$$

(d) YES. Because the eigenvalues of e^A are $e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3}$, $\det(e^A) = e^{\lambda_1}e^{\lambda_2}e^{\lambda_3} = e^{\lambda_1+\lambda_2+\lambda_3} = e^{\text{trace}(A)}$.

(e) Since orthogonal matrices do not change the length of **x**,

$\|A\mathbf{x}\| = \|Q\Lambda Q^T\mathbf{x}\| = \|\Lambda\mathbf{y}\|$, where we have defined $\mathbf{y} = Q^T\mathbf{x}$. Note also that

$\|\mathbf{y}\| = \|Q^T\mathbf{x}\| = \|\mathbf{x}\| = 1$. Therefore, the maximum value of

$\|\Lambda\mathbf{y}\| = \sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2 + \lambda_3^2 y_3^2}$ subject to unit vector **y** ($y_1^2 + y_2^2 + y_3^2 = 1$) is

determined by the maximum absolute eigenvalue of *A*, i.e., 3.

(f) $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T Q\Lambda Q^T\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y} = -3y_1^2 - 2y_2^2 + y_1^2 \geq -3.$

(g) *A* is nonsingular since all its eigenvalues are nonzero. *A* is also symmetric since it takes the form $A = Q\Lambda Q^T$. *A* is not orthogonal since $A^T A \neq I$. *A* is diagonalizable since it has independent eigenvectors. *A* is not positive definite since it has negative eigenvalues.

3.

(a) FALSE. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. A^k and *A* have different eigenvalues for $k > 1$.

(b) TRUE. The exponential e^A is symmetric alright, and has positive eigenvalues since $e^\lambda > 0$, where λ is the real eigenvalue of *A*.

(c) FALSE. For example, $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. The eigenvalues of *A* are not 1, 1.

(d) FALSE. *A* is positive definite if and only if all the upper-left submatrices have positive determinants. Here is a counterexample $A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$.

(e) FALSE. Write $\det A = \det(-A^T) = (-1)^n \det A$. If n is an even number, A may not be singular, for example, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

4.

(a) We require that $a > 0$ and $\det A = a^2 - 1 > 0$. Thus, $a > 1$.

(b) It is required that $\mathrm{Re}(\lambda) < 0$. Since the eigenvalues of A are $a+1, a-1$, we must have $a+1 < 0$ and $a-1 < 0$ Hence, $a < -1$.

(c) Clearly, the eigenvalues of A must be ± 1, and thus $a = 0$.

5.

(a) Solving $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ yields $X = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, indicating $[\mathbf{v}_1]_w = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, [\mathbf{v}_2]_w = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(b) Let $V = [\mathbf{v}_1 \quad \mathbf{v}_2]$. Then, $A = VBV^{-1}$. Clearly, $B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$.

(c) Let $W = [\mathbf{w}_1 \quad \mathbf{w}_2]$. We have $A = VBV^{-1}$ and $A = WCW^{-1}$. Write $B = V^{-1}AV = V^{-1}WCW^{-1}V$. Hence, $D = V^{-1}W = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$.