Matrix Theory **Exam 1**

10/25/2010

1. (35%) Answer true or false. No need to give a reason.

(a) (7%) $\left\{(x, y) \mid x^2 = y^2,\ x, y \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^2$.

(b) (7%) Let $V = \left\{(a,b,c,d) \mid a+b+c+d=0\right\}$, and

$W = \left\{(a,b,c,d) \mid a=b, c=d\right\}$. The dimension of $V \cap W$ is 1.

(c) (7%) Let V and W be two subspaces in $\mathbb{R}^7$. If $\dim V = \dim W = k$, and $\dim V \cap W = 2$, then $k \le 5$.

(d) (7%) If $(A+I)(B+I) = I$, then $AB = BA$.

(e) (7%) Let A and B be $n \times n$ matrices. We must have $\text{rank}\begin{bmatrix} A \\ B \end{bmatrix} = \text{rank}\begin{bmatrix} A & B \end{bmatrix}$.

2. (65%) Answer the following questions with explanation.

(a) (10%) What is the maximum value of $\text{rank}A$?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{bmatrix}$$

(b) (10%) Find the inverse of $B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}$, where A is an $n \times n$ invertible matrix and $\mathbf{a} \in \mathbb{R}^n$.

(c) (10%) Let A and B be $n \times n$ matrices. If $AB = 0$, show that $\text{rank}A + \text{rank}B \le n$.

(d) (10%) Let A be 3×2 and B be 2×5 matrices. What are the possible dimensions of $N(AB)$?

(e) (10%) Suppose $\mathbf{u} \in \mathbb{R}^n$. Determine the inverse of $I_n + \mathbf{u}\mathbf{u}^T$.

(f) (15%) Consider the following linear transformation T from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}: T(X) = X\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}X$. Find the matrix representation of T with respect to the ordered basis $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

What is the rank of T?