Matrix Theory
Solutions to Exam1 2010

1.

(a) FALSE. Suppose $x \neq 0$. Consider $\begin{bmatrix} x \\ -x \end{bmatrix} + \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix}$.

(b) TRUE. The intersection $V \cap W = \{(a,b,c,d) \mid a+b+c+d=0, a=b, c=d\}$

has only one free variable, and $V \cap W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$.

(c) FALSE. Since $\dim(V+W) = \dim V + \dim W - \dim(V \cap W) = 2k - 2 \leq 7$, it follows that $2k \leq 9$. Because k is a nonnegative integer, $k \leq 4$. Thus, k can never be 5.

(d) TRUE. Note that $(A+I)(B+I) = I$ implies $AB = -(A+B)$, and $(B+I)(A+I) = I$ implies $BA = -(A+B)$. Thus, $AB = BA$.

(e) FALSE. Suppose $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then,

$$\text{rank}\begin{bmatrix} A \\ B \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = 2 \quad \text{and} \quad \text{rank}\begin{bmatrix} A & B \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = 1.$$

2.

(a) The last three columns of A must be linearly dependent, so the maximum number of linearly independent columns, i.e., $\text{rank}A$, is 4. You can also reorder the columns of A and then apply row operations to reduce it to echelon form, as follows:

$$\begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{11} & a_{12} \\ a_{23} & a_{24} & a_{25} & a_{21} & a_{22} \\ 0 & 0 & 0 & a_{31} & a_{32} \\ 0 & 0 & 0 & a_{41} & a_{42} \\ 0 & 0 & 0 & a_{51} & a_{52} \end{bmatrix} \to \begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{11} & a_{12} \\ 0 & * & * & * & * \\ 0 & 0 & 0 & a_{31} & a_{32} \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

(b) Let $B^{-1} = \begin{bmatrix} X & \mathbf{y} \\ \mathbf{z}^T & w \end{bmatrix}$. The equation $B^{-1}B = \begin{bmatrix} X & \mathbf{y} \\ \mathbf{z}^T & w \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$

can be separated into the following four equations:

$XA + \mathbf{y}\mathbf{a}^T = I,$

$\mathbf{y} = \mathbf{0},$

$\mathbf{z}^T A + w\mathbf{a}^T = \mathbf{0}^T,$

$w = 1.$

Solving these equations yields $B^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -\mathbf{a}^T A^{-1} & 1 \end{bmatrix}.$

(c) The equation $AB = 0$ implies $C(B) \subseteq N(A),$ and thus $\dim C(B) \le \dim N(A).$ From the rank-nullity theorem, we have $\text{rank}A + \dim N(A) = n,$ and thus $\text{rank}A + \text{rank}B = \text{rank}A + \dim C(B) \le \text{rank}A + \dim N(A) = n.$

(d) Since $\text{rank}A \le 2$ and $\text{rank}B \le 2,$ it follows that $0 \le \text{rank}(AB) \le 2.$ Note that AB is $3\times 5.$ Using the rank-nullity theorem, we obtain $0 \le 5 - \dim N(AB) \le 2,$ i.e., $3 \le \dim N(AB) \le 5.$ Therefore, the possible dimensions of $N(AB)$ are 3, 4, 5.

(e) It is suggestive that $\left(I_n + \mathbf{u}\mathbf{u}^T\right)^{-1} = I + k\mathbf{u}\mathbf{u}^T.$ Thus, we have

$$\left(I + \mathbf{u}\mathbf{u}^T\right)\left(I + k\mathbf{u}\mathbf{u}^T\right) = I + \mathbf{u}\mathbf{u}^T + k\mathbf{u}\mathbf{u}^T + k\mathbf{u}\left(\mathbf{u}^T\mathbf{u}\right)\mathbf{u}^T = I + \left(1 + k + k\mathbf{u}^T\mathbf{u}\right) = I,$$

requiring $k = -\dfrac{1}{1+\mathbf{u}^T\mathbf{u}},$ where $\mathbf{u}^T\mathbf{u} \ge 0.$ So,

$$\left(I + \mathbf{u}\mathbf{u}^T\right)^{-1} = I - \frac{1}{1+\mathbf{u}^T\mathbf{u}}\mathbf{u}\mathbf{u}^T.$$

(f) Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$ Then,

$$T(X) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} - \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} = \begin{bmatrix} -2c & 2a-2d \\ 0 & 2c \end{bmatrix}.$$

Express the $T(X)$ with respect to the given basis:

$$\left[T(X)\right]_B = \begin{bmatrix} -2c \\ 2a-2d \\ 0 \\ 2c \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = [T]_B [X]_B,$$ which gives

the matrix representation of T. It is clear that $\text{rank}T = 2,$ since $[T]_B$ has two linearly independent columns.