Matrix Theory **Exam 2**

12/6/2010

1. (35%) Answer true or false. No need to give a reason.

   (a) (7%) $\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}$ are similar.

   (b) (7%) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is diagonalizable.

   (c) (7%) If Q is a real orthogonal matrix, i.e, $Q^T Q = I$, then the eigenvalues of Q must be either 1 or -1.

   (d) (7%) If A is similar to B and C is similar to D, then $A \oplus C$ is similar to $B \oplus D$.

   (e) (7%) $\operatorname{rank}\begin{bmatrix} A & 0 \\ B & I_n \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & 0 \\ I_n & B \end{bmatrix}$, where A and B are any $n \times n$ matrices.

2. (65%) Answer the following questions with explanation.

   (a) (15%) Let the Jordan form of A be

   $$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \oplus \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   What is the dimension of the nullspace of $A - 2I$? What is the trace of A^3?

   (b) (10%) Find $\begin{vmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{vmatrix}$.

   (c) (10%) Let A be a normal matrix, i.e., $A^* A = AA^*$. Show that if $A\mathbf{x} = \mathbf{0}$, then $A^*\mathbf{x} = \mathbf{0}$.

   (d) (10%) Suppose A is 2×2 and has eigenvalues $1, 2$. Use the Cayley-Hamilton theorem to express A^4 as a linear combination of A and I, i.e., find c and d such that $A^4 = cA + dI$.

(e) (10%) Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$. Write down the matrix exponential e^{At}.

(f) (10%) Consider the following linear transformation T from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$: $T(X) = X\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}X$. If a matrix $\hat{X}$ minimizes

$\left\| T(\hat{X}) - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\|$, what is $T(\hat{X})$? Note that

$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|^2 \equiv a^2 + b^2 + c^2 + d^2$.