Linear Algebra

Solutions to Exam 1 **2012**

1.

(a) Since P and L are invertible, it follows that A and U have the same nullspace. Here is the reasoning: if $A\mathbf{z}=\mathbf{0}$, then $U\mathbf{z}=L^{-1}PA\mathbf{z}=\mathbf{0}$; if $U\mathbf{z}=\mathbf{0}$, then $A\mathbf{z}=P^{-1}LU\mathbf{z}=\mathbf{0}$. For the given $\mathbf{x}$, we have $U\mathbf{x}\neq\mathbf{0}$, and thus $\mathbf{x}$ is not in the nullspace of A.

(b) Note that $C(A)$ and $C(U)$ are subspaces in $\mathbb{R}^4$. Since U has four pivots, rank(U) = rank(A)=4, implying that $C(U)=\mathbb{R}^4$ and $C(A)=\mathbb{R}^4$. Therefore, A and U have the same column space, which is $\mathbb{R}^4$.

(c) Let $(x_1,x_2,x_3,0,0)$ be a vector in the nullspace of A. Then, $x_1\mathbf{a}_1+x_2\mathbf{a}_2+x_3\mathbf{a}_3=\mathbf{0}$. However, $(x_1,x_2,x_3,0,0)$ is also in the nullspace of U, implying $x_1\begin{bmatrix}3\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}4\\2\\0\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\2\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$. Solving this equation yields $x_1=2\alpha, x_2=-\alpha, x_3=\alpha$. Thus, $\mathbf{a}_3=-2\mathbf{a}_1+\mathbf{a}_2$.

(d) Note that P is a permutation satisfying $P^4=I$. Then, $P^{99}\cdot P=P^{100}=(P^4)^{25}=I^{25}=I$. Thus, $P^{99}=P^{-1}=P^T$, i.e,

$$P^{99}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}.$$

2.

(a) Exchanging block rows of $\begin{bmatrix}A&I\\I&0\end{bmatrix}$, we get $\begin{bmatrix}I&0\\A&I\end{bmatrix}$. Then, eliminate block A and get the reduced row echelon form $\begin{bmatrix}I&0\\0&I\end{bmatrix}=I_{2n}$.

(b) Note that $\begin{bmatrix}I&B\\0&A\end{bmatrix}=\begin{bmatrix}I&0\\0&A\end{bmatrix}\begin{bmatrix}I&B\\0&I\end{bmatrix}=XY$. Then, $\begin{bmatrix}I&B\\0&A\end{bmatrix}^{-1}=Y^{-1}X^{-1}$.

Since A is invertible, it is easy to see that $X^{-1}=\begin{bmatrix}I&0\\0&A\end{bmatrix}^{-1}=\begin{bmatrix}I&0\\0&A^{-1}\end{bmatrix}$,

and $Y^{-1}=\begin{bmatrix}I&B\\0&I\end{bmatrix}^{-1}=\begin{bmatrix}I&-B\\0&I\end{bmatrix}$. Hence,

$$\begin{bmatrix} I & B \\ 0 & A \end{bmatrix}^{-1} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} I & -BA^{-1} \\ 0 & A^{-1} \end{bmatrix}.$$

3.

(a) TRUE. Note that $\operatorname{rank}(A) = \dim C(A) = 2$ and $\dim N(A) = 1$. Since A is 3 by 4,, the rank-nullity theorem states that $\operatorname{rank}(A) + \dim N(A) = 4$, which is a contradiction.

(b) TRUE. From the rank-nullity theorem, we know that $\operatorname{rank}(A) + \dim N(A) = 4$. However, $\operatorname{rank}(A) \le 3$, it follows that $\dim N(A) > 0$.

(c) FALSE. For example, $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon forms of A and B are $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, respectively.

(d) FALSE. When $\operatorname{rank}(A) = 3$, we have $C(A) = \mathbb{R}^3$, and thus $A\mathbf{x}=\mathbf{b}$ is solvable for every $\mathbf{b} \in \mathbb{R}^3$. This means that there exists B such that $AB^T = I_3$. For example, $AB^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(e) TRUE. Because rank(A) = rank(A^TA)=rank(0)=0, A has no pivots, i.e., A=0.