Linear Algebra
Solutions to Exam 1 **2013**

1.

(a) Let $EA=B$. Each row of E multiplies the whole matrix A. The result is a row of B. Since $\text{row}_1(A)+\text{row}_2(A)=(0,0,2,0,4)$ and $\text{row}_1(A)+\text{row}_2(A)+\text{row}_3(A)=(0,0,0,0,0)$, the third row of A is $(0,0,-2,0,-4)$.

(b) Because E is nonsingular, A and B have the same row space. The row space of A is spanned by $(1,1,4,2,5)$, $(0,0,2,0,4)$. Perform row operation to get an echelon form:

$$\begin{bmatrix}1&0&1\\1&0&1\\4&2&3\\2&0&2\\5&4&2\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&0&0\\0&2&-1\\0&0&0\\0&4&-3\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&0&0\\0&2&-1\\0&0&0\\0&0&-1\end{bmatrix}$$

From the last row, we know that $(1,1,3,2,2)$ is not in the row space of A.

(c) The reduced row echelon form of A is $\begin{bmatrix}1&1&0&2&-3\\0&0&1&0&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. So the nullspace matrix of A is $\begin{bmatrix}-1&-2&3\\1&0&0\\0&0&-2\\0&1&0\\0&0&1\end{bmatrix}$. The columns of the nullspace matrix form a basis for the nullspace of A: $\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}3\\0\\-2\\0\\1\end{bmatrix}$.

(d) The complete solution of $A^T\mathbf{y}=\mathbf{0}$ is the left nullspace of A, which is spanned by the third and fourth rows of E. Therefore, $\mathbf{y}=\alpha\begin{bmatrix}1\\1\\1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$.

(e) Given $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, we have $EA\mathbf{x} = B\mathbf{x} = E\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Since $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is not in the column space of B, the linear equation is not consistent.

2.

(a) Note that

$$\left(A(A+B)^{-1}B\right)^{-1} = B^{-1}(A+B)A^{-1} = B^{-1} + A^{-1}$$

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Thus, $A(A+B)^{-1}B = B(A+B)^{-1}A.$

(b) Suppose $\begin{bmatrix} A & 0 \\ B & A \end{bmatrix}\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Then,

$AX = I,\ BX + AY = 0,\ AZ = I.$ Solving yields

$X = Z = A^{-1},\ Y = -A^{-1}BA^{-1}.$ Therefore, $\begin{bmatrix} A & 0 \\ B & A \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -A^{-1}BA^{-1} & A^{-1} \end{bmatrix}.$

3.

(a) FALSE. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$ if and only if $A\mathbf{x} = \mathbf{0}$ has only trivial solution $\mathbf{x} = \mathbf{0}$. If A has independent columns, then the nullspace of A is $\{\mathbf{0}\}$. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) TRUE. Since $\operatorname{rank}(A) \le 3$, $\operatorname{rank}(A^T A) = \operatorname{rank}(A) \le 3.$ It follows that every 4 by 4 matrix $A^T A$ must be singular.

(c) TRUE. The statement is equivalent to "if A^T and B^T have the same reduced row echelon form, then A^T and B^T have the same row space."

(d) TRUE. Clearly, A is m by 3, where $m \ge \operatorname{rank}(A) = 2$. The reduced row echelon form of A is $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$. However, $R\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. This means that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the nullspace of A.