Linear Algebra

Solutions to Exam 1 **2015**

1.

(a) Since E is an invertible matrix, A is row equivalent to $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus, $\text{rank}(A) = \text{rank}(B) = 2.$

(b) YES. Note that the row spaces of A and B are identical. Add (1,1,1) and reduce $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 1 & 1 & 1 \end{bmatrix}$ to echelon form: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$. This indicates that the span of (1,2,3), (0,-3,-6), and (1,1,1) equals the span of (1,2,3) and (0,-3,-6). Therefore, (1,1,1) is in the row space of *A*.

(c) YES. Note that $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in the column space of A when $A\mathbf{x} = \mathbf{b}$ is solvable. Since $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ satisfies $b_3 - 2b_2 + b_1 = 0$, it is in the column space of *A*.

(d) From $E\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 4b_1 \\ b_3 - 2b_2 + b_1 \end{bmatrix}$, it can be concluded that $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$.

(e) The reduced row echelon form of A is $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the nullspace of A is spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, meaning that the complete solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x}_h = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(f) From (d), we know that $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. Hence, a basis for the left nullspace of A is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

2.

(a) The subspace of all vectors whose components are equal is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. So, a basis is for the subspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in the subspace of all vectors whose components add to zero, then $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$. Thus, the subspace is simply the nullspace of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, which has a basis as follows: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

3.

(a) TRUE. If the rank of $\left[A \,|\, \mathbf{b} \right]$ is equal to the rank of A, then $\mathbf{b}$ is in the column space of A, and thus $A\mathbf{x} = \mathbf{b}$ is solvable.

(b) TRUE. If AA^T is invertible, then $\text{rank}(A) = \text{rank}(AA^T) = m$. Since

$\operatorname{rank}(A) \le n$, it follows that $m \le n$.

(c) FALSE. If there are some $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has at no solution, then $\mathbf{b}$ is not in the column space of A, implying $\operatorname{rank}(A) < m$. However, it is possible that $m < n$. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(d) TRUE. If $A\mathbf{x} = \mathbf{b}$ has infinite solutions for every $\mathbf{b}$, then $\operatorname{rank}(A) = m$ and $\operatorname{rank}(A) < n$. Therefore, $m < n$.