Linear Algebra

Solutions to Exam 1 **2016**

1.

(a) Since E is an invertible matrix, A is row equivalent

to $B = EA = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus, $\dim C(A) = \dim C(B) = 2$.

(b) Note that the left nullspace of A is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Thus, $\dim N(A^T) = 1$.

(c) YES. Note that $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is reduced to $B\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$.

(d) YES. The row space of A is spanned by $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. It follows that

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(e) If $c\mathbf{a}_2 + \mathbf{a}_3 = 0$, then $E(c\mathbf{a}_2 + \mathbf{a}_3) = c(E\mathbf{a}_2) + E\mathbf{a}_3 = \mathbf{0}$. That is,

$c\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence, $c = \frac{1}{2}$.

(f) $A\mathbf{x}=\mathbf{b}$ is consistent if and only $B\mathbf{x}=EA\mathbf{x}=E\mathbf{b}$ is consistent. From the zero row of B, $b_1 + b_2 = 0$ ensures that $B\mathbf{x}=E\mathbf{b}$ is solvable.

(g) Since $\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the nullspace matrix of A is given by

$$N=\begin{bmatrix}-2 & 1\\ 1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}.$$ Therefore, a basis for the nullspace of A is $\left\{\begin{bmatrix}-2\\ 1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 1\\ 0\end{bmatrix}\right\}$.

(h) Note that $E\mathbf{a}_1=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$. Using row operations, we have

$$\begin{bmatrix}3 & 2 & 4 & 1\\ 2 & 1 & 3 & 0\\ 1 & 1 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & -3\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 1\end{bmatrix}$$

Thus, $\mathbf{a}_1=\begin{bmatrix}-3\\ 3\\ 1\end{bmatrix}$.

2.

(a) FALSE. For example, **u**=**v** and {**u**,**w**} is linearly independent.

(b) TRUE. Note that any two columns of A can be pivot columns. There are $C_2^4=6$ types of 3 by 4 matrices in reduced row echelon form.

(c) FALSE. For example, $A=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

(d) FALSE. For example, $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$. It is impossible to find a matrix B such that $A^TB^T=I_3$, because $A^T\mathbf{y}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$ is inconsistent.

(e) TRUE. We can show that $N(B)\subset N(AB)$ and $N(AB)\subset N(B)$. If **x** is in $N(B)$, then B**x**=0, and thus AB**x**=**0**. If **x** is in $N(AB)$, then AB**x**=0, and thus $A^{-1}AB$**x**= B**x=0**.

(f) FALSE. Note that $\dim C(A)=\dim C(A^T)$ and $\dim C(A^T)+\dim N(A)=3$. Thus, $\dim C(A)+\dim N(A)=3$. If $C(A)=N(A)$, then $\dim C(A)+\dim N(A)$ is even. We have a contradiction.