Linear Algebra **Exam 2**

5/14/2015

1. (36%) Let $A=\begin{bmatrix} a & 0 & 0 & 2 \\ 2 & 2 & -2 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix}$. It is known that the orthogonal projection matrix onto the row space of A is $P=\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

   (a) (8%) What is the rank of A?

   (b) (8%) Are you sure that $AP = A$? Give a reason.

   (c) (10%) What is the closest vector in the nullspace of A to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

   (d) (10%) Find an orthonormal basis for the column space of A.

2. (38%) Answer the following questions.

   (a) (10%) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and $\beta = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. If $A=\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ is the matrix representation of T with respect to β, find the matrix representation of T with respect to the standard basis of $\mathbb{R}^2$.

   (b) (10%) Let $T:V \to V$ be a linear transformation, $\dim V = 3$, and $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors in V. It is known that $T(\mathbf{u}) = \mathbf{u}+2\mathbf{v}+\mathbf{w}$, $T(\mathbf{v}) = \mathbf{v}+\mathbf{w}$, $T(\mathbf{w}) = \mathbf{u}+\mathbf{v}$. Find the kernel of T, i.e., $N(T) = \{\mathbf{x}\in V \mid T(\mathbf{x}) = \mathbf{0}\}$. Express $N(T)$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   (c) (10%) Let $T(x, y) = (x+3y, 2x+4y)$. Find the area of $T(S)$, where $S = \{(x, y) \mid -1 \le x \le 2,\ 3 \le y \le 5\}$.

   (d) (8%) True or false: Suppose $T:\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation. If T maps $\mathbb{R}^3$ onto $\mathbb{R}^3$, then T is one-to-one. If yes, state your reason; otherwise, give a counterexample.

3. (26%) Answer the following questions.

(a) (8%) Let $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & a \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 1 \\ 6 & 5 & 2 & 0 \\ 7 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Find a so that $\det A = 20$.

(b) (8%) Let Q be an n by n orthogonal matrix and C be the cofactor matrix of Q. Show that C is also an orthogonal matrix.

(c) (10%) Let Q be a 3 by 2 matrix having orthonormal columns. Find $\det(Q^T Q)$ and $\det(QQ^T)$.