Linear Algebra **Exam 2**

5/12/2016

1. (30%) Let $Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix}$ be a 3×3 orthogonal matrix such that $\mathbf{q}_3 = \mathbf{q}_1 \times \mathbf{q}_2$, where $\times$ denotes the cross product. Consider the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{x}\times\mathbf{q}_1 + \left(\mathbf{q}_2^T\mathbf{x}\right)\mathbf{q}_1$.
   (a) (10%) What is the rank of *T*?
   (b) (10%) Find *k* so that the linear system $T\left(\mathbf{x}\right) = \mathbf{q}_1 + 2\mathbf{q}_2 + k\mathbf{q}_3$ is consistent and then find all the solutions in terms of $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$.
   (c) (10%) What is the closest vector in the range of *T* to $\mathbf{b} = \mathbf{q}_1 + 2\mathbf{q}_2 + 3\mathbf{q}_3$?

2. (48%) Let *S* be a subspace in $\mathbb{R}^4$ spanned by (1,0,1,1) and (1,1,0,1).
   (a) (10%) Find an orthonormal basis of $S^{\perp}$.
   (b) (10%) If $\mathbf{v} = (5,5,5,5)$, find **x** in *S* and **y** in $S^{\perp}$ so that $\mathbf{v} = \mathbf{x} + \mathbf{y}$.
   (c) (10%) Find a matrix *M* so that $C(M) = S^{\perp}$ and $N(M) = S$.
   (d) (8%) Let *P* be the orthogonal projection matrix onto the subspace *S*. Is it true that $\operatorname{rank}(P) + \operatorname{rank}(I-P) = 4$? Explain.
   (e) (10%) Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Find the minimal solution of $A\mathbf{x} = \mathbf{b}$, i.e., the solution has the least value of $\|\mathbf{x}\|$.

3. (22%) The following matrices are real.
   (a) (6%) True or false: If *A* is an orthogonal matrix, then $\det(-A^2) = -1$. Give a reason or counterexample.
   (b) (6%) True or false: If *A* is an orthogonal projection matrix and *A* is nonsingular, then $\det A = 1$. Give a reason or counterexample.
   (c) (10%) If *A*, *B*, and *C* are $n\times n$ matrices and *A* is nonsingular, show that

$$\det\begin{bmatrix} A & B \\ I_n & C \end{bmatrix} = \det(AC - B).$$