Linear Algebra **Exam 2**

5/17/2012

1.

(a) Find the images of standard basis vectors as follows:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=1+t+t^2,$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)-T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=(1+t^2)-(1+t+t^2)=-t,$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)-T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=(3+t+3t^2)-(1+t+t^2)=2+2t^2.$$

Thus, the matrix representation of T with respect to the standard basis is

$A=\begin{bmatrix}1&0&2\\1&-1&0\\1&0&2\end{bmatrix}$. The kernel of T is the nullspace of A, which is spanned by $\begin{bmatrix}-2\\-2\\1\end{bmatrix}$.

(b) Define the basis matrices $B=\begin{bmatrix}1&1\\1&2\end{bmatrix}$ and $C=\begin{bmatrix}3&2\\1&1\end{bmatrix}$. For any $\mathbf{x}\in\mathbb{R}^2$, the coordinates vectors are given by $[\mathbf{x}]_B=B^{-1}\mathbf{x}$ and $[\mathbf{x}]_C=C^{-1}\mathbf{x}$. Let X be the change-of-coordinates matrix from basis C to basis B. Then, $[\mathbf{x}]_B=X[\mathbf{x}]_C$, which leads to $X=B^{-1}C=\begin{bmatrix}5&3\\-2&-1\end{bmatrix}$. Therefore,

$$[T]_C=X^{-1}[T]_B X=\begin{bmatrix}-1&-3\\2&5\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}5&3\\-2&-1\end{bmatrix}=\begin{bmatrix}7&3\\-10&-4\end{bmatrix}.$$

(c) Write the matrix representation and apply row operations to get

$$\begin{bmatrix}1&-1&0&1\\0&1&-2&0\\2&-1&-2&1\end{bmatrix}\to\begin{bmatrix}1&-1&0&1\\0&1&-2&0\\0&1&-2&-1\end{bmatrix}\to\begin{bmatrix}1&-1&0&1\\0&1&-2&0\\0&0&0&-1\end{bmatrix}.$$

The echelon form has three pivots indicating that the three polynomials are linearly independent.

(d) Since T is one-to-one, we know that $N(T)=\{\mathbf{0}\}$. Since T does not map onto $\mathbb{R}^m$, it follows that $r < m$. According to the rank-nullity theorem, we obtain $n = r + \dim N(T) = r + 0 = r$. Thus, $r = n < m$.

2.

(a) Since $\operatorname{rank}(A) = 1$, we know that A is row equivalent to $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, the nullspace of A is spanned by $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Apply Gram-Schmidt orthogonalization process, we get $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Hence, an orthonormal basis for the nullspace of A is $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(b) Let Q be the projection matrix onto the row space of A. Since the row space of A is spanned by $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, we get $Q = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Because the nullspace is the orthogonal complement of the row space, the projection matrix onto the nullspace of A is given by $P = I - Q = \frac{1}{6}\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$.

(c) The closest vector in the row space of A to the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the orthogonal projection of $\mathbf{b}$ onto the row space, which is $Q\mathbf{b} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a} = \frac{4}{6}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

(d) YES. Because P is the projection matrix onto the nullspace of A, it follows that $C(P) = N(A)$.

3.

(a) Let K denote the given skew-symmetric matrix, $K^T = -K$. Then, $\det K = \det K^T = \det(-K) = (-1)^n \det K$. Since n=5, we have $\det K = -\det K$, implying that $\det K = 0$.

(b) Write $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and compute $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$. The area of the parallelogram is given by $\sqrt{\det(A^T A)} = \sqrt{6}$.

(c) FALSE. If $A = QR$, then $\det A = \det(QR) = (\det Q)(\det R)$. However, $\det Q = \pm 1$, because $Q^T Q = I$ yields $\det(Q^T Q) = (\det Q^T)(\det Q) = (\det Q)^2 = \det I = 1$. For example, $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = QR$. Note that $\det A = -1$, but $\det R = 1$.