Linear Algebra

Solutions to Exam 2 **2013**

1.

(a) Rewrite A as follows: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix} = QB,$

where Q has orthonormal columns, implying $Q^T Q = I$. The normal equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ satisfied by the least-squares solution $\hat{\mathbf{x}}$ becomes

$$(QB)^T (QB)\hat{\mathbf{x}} = (QB)^T \mathbf{b}$$
$$\Rightarrow B^T Q^T QB\hat{\mathbf{x}} = B^T Q^T \mathbf{b}$$
$$\Rightarrow Q^T QB\hat{\mathbf{x}} = Q^T \mathbf{b}$$
$$\Rightarrow B\hat{\mathbf{x}} = Q^T \mathbf{b}$$

That is, $\begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Solving yields $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/2 \end{bmatrix}$.

(b) Let P be the orthogonal projection matrix onto the column space of A. Since $C(A)=C(Q)$, P is the orthogonal projection matrix onto the column space of Q. Because Q has orthonormal columns, it follows that

$P = QQ^T = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Since the closest vector in $C(A)$ to $\mathbf{z}$ is the orthogonal projection of $\mathbf{z}$ onto $C(A)$, our goal is to solve $P\mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. The general solution is given by $\mathbf{z} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

(c) The nullspace of A^T is the orthogonal complement of the column space of A. Therefore, the orthogonal projection matrix onto $N(A^T)$ is

$$I-P=\frac{1}{2}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}.$$

(d) True. Because $N(A^T)=N(AA^T)$, M is also the orthogonal projection matrix onto the nullspace of A^T. Since the column space of A is the orthogonal complement of $N(A^T)$, the orthogonal projection of every column of A onto $N(AA^T)$ must be zero. Therefore, $MA=0$.

2.

(a) The standard matrix of T is $A=\begin{bmatrix}4 & -2\\ -5 & 3\end{bmatrix}$. The area of $T(S)$ is the product of the area of S and $|\det A|$. The area of S is 2 and $|\det A|=2$. Thus, the area of $T(S)$ is 4.

(b) Let $B=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$. Note that $\mathbf{x}=B[\mathbf{x}]_\beta$ and $[T(\mathbf{x})]_\beta=A[\mathbf{x}]_\beta$. It follows that $T(\mathbf{x})=B[T(\mathbf{x})]_\beta=BA[\mathbf{x}]_\beta=BAB^{-1}\mathbf{x}$. Hence,

$$T(2,5)=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}2\\ 5\end{bmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix}.$$

(c) True. Because $\{p(t),q(t),r(t)\}$ is linearly independent,

$c_1p(t)+c_2q(t)+c_3r(t)=0$ implies $c_1=c_2=c_3=0$. Thus,

$d_1(p(t)+q(t))+d_2(q(t)+r(t))+d_3(p(t)+r(t))$

$=(d_1+d_3)p(t)+(d_1+d_2)q(t)+(d_2+d_3)r(t)=0$

implies $d_1+d_3=d_1+d_2=d_2+d_3=0$. Solving yields $d_1=d_2=d_3=0$.

This proves that $\{p(t)+q(t),\ q(t)+r(t),\ p(t)+r(t)\}$ is linearly independent.

(d) Since A is 6 by 3 and is one-to-one, $\dim C(A)=3-\dim N(A)=3-0=3$. Because B maps $\mathbb{R}^5$ onto $\mathbb{R}^3$, the range (column space) of B is $\mathbb{R}^3$. This means that $C(AB)=C(A)$. Therefore,

$\operatorname{rank}(AB)=\dim C(AB)=\dim C(A)=3$.

3.

(a) $$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 3 \\ 0 & 0 & 5 & 3 & 5 \\ 5 & 3 & 9 & 8 & 4 \\ 8 & 5 & 4 & 6 & 3 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 3 & 0 \\ 0 & 5 & 3 & 5 & 0 \\ 5 & 9 & 8 & 4 & 3 \\ 8 & 4 & 6 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 0 & 0 \\ 5 & 3 & 5 & 0 & 0 \\ 5 & 8 & 4 & 5 & 3 \\ 8 & 6 & 3 & 8 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \\ 5 & 3 & 5 \end{vmatrix} \cdot \begin{vmatrix} 5 & 3 \\ 8 & 5 \end{vmatrix} = 1 \cdot 1 = 1$$

(b) If all entries of A and A^{-1} are integers, then $\det A$ and $\det A^{-1}$ are nonzero integers. However, $\det A^{-1} = \dfrac{1}{\det A}$. So we must have $\det A = 1$ or -1.

(c) Recall that $AC^T = (\det A)I_n$. Taking determinant,

$\det(AC^T) = (\det A)(\det C^T) = (\det A)^n(\det I_n) = (\det A)^n$. We obtain

$\det C = \det C^T = (\det A)^{n-1}$. If A is singular, it follows that $\det A = 0$, and thus $\det C = 0$. Therefore, C is singular.