Linear Algebra

Solutions to Exam 2 **2015**

1.

(a) Since $C(P) = C(A^T)$, $\text{rank}(A) = \dim C(A^T) = \dim C(P) = 2.$

(b) YES. Rewrite $AP = A$ as $A(I-P) = 0.$ Note that $I - P$ is the orthogonal projection matrix onto the nullspace of *A*, because the nullspace is the orthogonal complement of the row space of *A*. It follows that $(I-P)\mathbf{x}$ is in the nullspace of *A*. Hence, $A(I-P)\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$ implies $A(I-P) = 0.$

(c) The closest vector in the nullspace of *A* to $\mathbf{b} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is the projection of $\mathbf{b}$ onto the nullspace of *A*:

$$(I-P)\mathbf{b} = \mathbf{b} - P\mathbf{b} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}.$$

(d) Since $\text{rank}(A) = 2,$ it is clear that $\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{\begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$ is a basis for the column space of *A*. Use Gram-Schmidt orthogonalization process to obtain an orthonormal basis as follows:

Set $\mathbf{u}_1 = \mathbf{a}_1 = \begin{bmatrix}0\\2\\1\end{bmatrix}$. Compute $\mathbf{u}_2 = \mathbf{a}_2 - \frac{\mathbf{u}_1^T\mathbf{a}_2}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix}2\\2\\1\end{bmatrix} - \frac{5}{5}\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$. After normalization, we obtain an orthonormal basis $\left\{\frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$.

2.

(a) Let $B = \begin{bmatrix}2 & 3\\1 & 1\end{bmatrix}$. Note that $\mathbf{x} = B[\mathbf{x}]_\beta$ and $[T(\mathbf{x})]_\beta = A[\mathbf{x}]_\beta$. It follows that $T(\mathbf{x}) = B[T(\mathbf{x})]_\beta = BA[\mathbf{x}]_\beta = BAB^{-1}\mathbf{x}.$ Hence, the matrix representation of T with respect to the standard basis of $\mathbb{R}^2$ is

$$BAB^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -10 \\ 2 & -2 \end{bmatrix}.$$

(b) Let $\beta = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for *V*. The matrix representation of *T* with respect to β is

$$[T]_\beta = \left[[T(\mathbf{u})]_\beta \quad [T(\mathbf{v})]_\beta \quad [T(\mathbf{w})]_\beta \right] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The reduced row echelon form of $[T]_\beta$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. It follows immediately that the nullspace of $[T]_\beta$ is spanned by $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. In other words, the kernel of *T* is spanned by $-\mathbf{u} + \mathbf{v} + \mathbf{w}$.

(c) Let *A* be the matrix representation of *T* with respect to the standard basis. It is easy to see $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Note that the area of *T*(*S*) is the product of the area of *S* and $|\det A|$. The area of *S* is $3 \cdot 2 = 6$ and $|\det A| = 2$. Thus, the area of *T*(*S*) is 12.

(d) TRUE. Since *T* maps $\mathbb{R}^3$ onto $\mathbb{R}^3$, $\operatorname{rank}(T) = \dim R(T) = 3$. The rank-nullity theorem states that $\operatorname{rank}(T) = 3 - \dim\ker(T)$. Thus, $\dim\ker(T) = 0$, implying that *T* is one-to-one.

3.

(a) Note that

$$\begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & a \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & a \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & a \end{vmatrix} = 6 - a$$

$$\begin{vmatrix} 2 & 3 & 4 & 1 \\ 6 & 5 & 2 & 0 \\ 7 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 7 & 2 & 0 & 0 \\ 6 & 5 & 2 & 0 \\ 2 & 3 & 4 & 1 \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot 1 = 4.$$

From $\det A = (6-a)\cdot 4 = 20$, we obtain $a = 1$.

(b) Recall that Q is an orthogonal matrix if and only if $Q^{-1} = Q^T$. Taking determinant, we obtain $\det(Q^{-1}) = (\det Q)^{-1}$ and $\det Q^T = \det Q$. Thus, $(\det Q)^{-1} = \det Q$, i.e., $\det Q = \pm 1$. Since $QC^T = (\det Q)I$, we obtain $C^T = (\det Q)Q^{-1} = (\det Q)Q^T$. Thus, $C = (\det Q)Q$ and $C^{-1} = (\det Q)^{-1}Q^{-1} = (\det Q)^{-1}Q^T$. Hence, $C^{-1} = C^T$.

(c) Note that $Q^TQ = I_2$ and QQ^T is a 3 by 3 matrix with $\text{rank}(QQ^T) = 2$.

Therefore, $\det(Q^TQ) = 1$ and $\det(QQ^T) = 0$.