Linear Algebra **Final Exam**

6/19/2012

You must show your work to get credit.

1. (35%) Suppose A is a 3 by 3 real matrix having eigenvalues $0, -2$, 3, and the corresponding eigenvectors are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Let P be the projection matrix onto the column space of A. Answer the following questions:

   (a) (3%) What is the rank of A ?

   (b) (3%) What is the trace of A^3 ?

   (c) (3%) What is the determinant of $(A+I)^{-1}$?

   (d) (3%) What are the eigenvalues of P?

   (e) (3%) Find all the solutions to $A\mathbf{x} = \mathbf{v} + \mathbf{w}$.

   True or false. State your reason briefly or give a counterexample.

   (f) (4%) A is similar to $B = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

   (g) (4%) $A\mathbf{x} = \mathbf{u}$ has no solution.

   (h) (4%) The row space of A is spanned by $\mathbf{v}$ and $\mathbf{w}$.

   (i) (4%) P is orthogonally diagonalizable, i.e., there exists an orthonormal matrix Q such that $P = QDQ^T$, where D is a diagonal matrix.

   (j) (4%) The eigenvalues of $A^T A$ are 0, 4, 9.

2. (20%) Let $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

   (a) (5%) Find all matrices R so that $R^2 = A$.

   (b) (5%) Write down the 2 by 2 matrix e^A.

   (c) (5%) Find c and d so that $A^3 = cA + dI$. (Hint: Use Cayley-Hamilton theorem.)

   (d) (5%) Find the unit vector $\mathbf{x}$, $\|\mathbf{x}\| = 1$, at which $\mathbf{x}^T A\mathbf{x}$ is maximized. What is the maximum value of $\mathbf{x}^T A\mathbf{x}$?

3. (15%) Let $A = \begin{bmatrix} 0 & 5 \\ -2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real, and $b \geq 0$.

(a) (5%) Find an invertible matrix S such that $C = S^{-1}AS$. What are a and b?

(b) (5%) If $\|C\mathbf{x}\| = \|\mathbf{x}\|$ for every nonzero $\mathbf{x} \in \mathbb{R}^2$, what can you tell about a and b?

(c) (5%) Suppose C^k converges to the zero matrix as $k \to \infty$. What can you tell about a and b?

4. (15%) Let $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$, where c is a real number. Note that A can be expressed as $A = (c-1)I + E$, where $E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{e}\mathbf{e}^T$ is a rank-one matrix and $\mathbf{e}^T = (1,1,1)$.

(a) (5%) Find the eigenvalues and the corresponding orthonormal eigenvectors of E. (Hint: You can find the eigenvectors by inspection.)

(b) (5%) Find the eigenvalues and the corresponding orthonormal eigenvectors of A.

(c) (5%) What number c makes A nonsingular? Express the inverse of A in terms of I and E.

5. (15%) Answer the following questions. Let A be an n by n real matrix.

(a) (5%) If A is similar to B, show that $A + I$ is similar to $B + I$.

(b) (5%) If A is invertible, show that AB is similar to BA.

(c) (5%) If $A^T = -A$, show that $\mathbf{x}^T A\mathbf{x} = 0$ for every real vector $\mathbf{x}$.