Linear Algebra **Final Exam**

6/18/2013

You must show your work to get credit.

1. (30%) Suppose A is a 3 by 3 real matrix having eigenvalues $\alpha, \beta, \gamma,$ and the corresponding eigenvectors are **u**, **v**, **w**. In each case give all the information you can about the eigenvalues $\alpha, \beta, \gamma,$ when A has the following property:

   (a) (3%) $A^2 = 0$
   (b) (3%) $\det(A^2) = 0$
   (c) (3%) A is not diagonalizable.
   (d) (3%) A is symmetric positive definite.
   (e) (3%) The eigenvalues of $A - I$ are 1, 5, and 6.
   (f) (3%) A is similar to $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & -1 & 5 \end{bmatrix}$.
   (g) (3%) A is an orthogonal projection matrix.
   (h) (3%) A is an orthogonal matrix.
   (i) (3%) A is symmetric and $\{\mathbf{u}, \mathbf{v}\}$ is a basis for the column space of A.
   (j) (3%) $(A+I)^k$ approaches the zero matrix as $k \to \infty$.

2. (50%) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

   (a) (5%) What is the matrix representation of A relative to the basis

   $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$?

   (b) (5%) Are you sure that AB is similar to BA, for every B? Why or why not.
   (c) (5%) Let $B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Find the conditions on a, b and c so that AB is symmetric.
   (d) (5%) Is $\begin{bmatrix} I & A \\ A & I \end{bmatrix}$ positive definite? State your reason.
   (e) (5%) Find a nonsingular matrix M so that $M^{-1}AM = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
   (f) (5%) Find nonzero matrices B and C so that $A = B + C$ and $BC = 0$.

(g) (5%) Find $\det(e^A)$.

(h) (5%) Let $B = A + kI$. Find the conditions on k so that $e^{Bt} \to 0$ as $t \to \infty$.

(i) (5%) Find the unit vector $\mathbf{x}$, $\|\mathbf{x}\| = 1$, at which $\mathbf{x}^T A\mathbf{x}$ is minimized. What is the minimum value of $\mathbf{x}^T A\mathbf{x}$?

(j) (5%) Find a nonsymmetric matrix B so that $\|AB\mathbf{x}\| = \|\mathbf{x}\|$, for every $\mathbf{x}$.

3. (20%) Answer the following questions.

(a) (5%) If we have $A\mathbf{x} = \lambda\mathbf{x}$, show that $A^T - \lambda I$ is singular.

(b) (5%) Let A and B be invertible matrices. If A is similar to B, show that A^{-1} is similar to B^{-1}.

(c) (5%) Let A be a real symmetric matrix. Explain why e^A is symmetric and positive definite.

(d) (5%) Let A be a 3 by 3 matrix. If $A^3 = -A^2 + 2A$, use Cayley-Hamilton theorem to find the three eigenvalues of A.