Linear Algebra **Final Exam**

6/23/2015

You must show your work in order to receive partial credit.

1. (30%) Suppose A is a 3 by 3 real matrix having eigenvalues $\alpha,\ \beta,\ \gamma,$ and the corresponding eigenvectors are **u**, **v**, **w**. In each case give all the information you can about the eigenvalues $\alpha,\ \beta,\ \gamma,$ when A has the following property:

   (a) (3%) The eigenvalues of $2A-3I$ are 1, 5, and 7.

   (b) (3%) $A^3 - A = 0$

   (c) (3%) A is symmetric.

   (d) (3%) $\det(A)=0$ and $\operatorname{trace}(A)>0$.

   (e) (3%) A is not diagonalizable and $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for the nullspace of A.

   (f) (3%) The rank of A is 1.

   (g) (3%) $A = M^{-1}BM$, where $B=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$.

   (h) (3%) A is a permutation matrix.

   (i) (3%) A is an orthogonal projection matrix onto the subspace spanned by the linearly independent set $\{\mathbf{u},\mathbf{v}\}$.

   (j) (3%) The dynamic system $\dfrac{d\mathbf{y}}{dt}=A\mathbf{y}$ is stable.

2. (40%) True or false. Give a reason or counterexample if false. All the following matrices are real.

   (a) (4%) If A is 3 by 3 and has eigenvalues 0, 0, 0, then A^2=0.

   (b) (4%) If A is 3 by 3 and has eigenvalues 0, 0, 1, then the rank of A is 1.

   (c) (4%) If A is diagonalizable, then $I+A+A^2$ is diagonalizable.

   (d) (4%) If A is nonsingular, then $AB+I$ is similar to $BA+I$.

   (e) (4%) It is impossible that A and B are not similar, but A^2 and B^2 are similar.

   (f) (4%) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

(g) (4%) If A is nonsingular and antisymmetric, i.e., $A^T = -A$, then $\det(A) > 0$.

(h) (4%) If A is symmetric and similar to B, then B is symmetric.

(i) (4%) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & 0 \end{bmatrix}$ is not positive definite.

(j) (4%) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$.

3. (30%) Answer the following questions.

(a) (6%) Find an invertible matrix M such that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = M^{-1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}M$.

(b) (6%) Find an invertible matrix M and a matrix of the form $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = \begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix} = MBM^{-1}$.

(c) (6%) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the matrix $\cos A$.

(d) (6%) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. Find the maximum value of $\|A\mathbf{x}\|$, where $\mathbf{x}$ is a unit vector.

(e) (6%) If 3 by 3 matrix A has eigenvalues 1, 1, 2, find a, b, c so that $A^{-1} = aA^2 + bA + cI$.