Linear Algebra **Final Exam**

6/14/2016

You must show your work in order to receive partial credit.

1. (20%) Suppose A is a 3 × 3 real symmetric matrix. In each case give all the information you can about the eigenvalues of A, when A has the following property:

   (a) (4%) The rank of A is 2.

   (b) (4%) A is a reflection matrix about some plane passing through the origin in $\mathbb{R}^3$.

   (c) (4%) A is an orthogonal projection matrix onto some plane passing through the origin in $\mathbb{R}^3$.

   (d) (4%) $A^2 + A - 2I = 0$ and $\text{trace}(A) = 0$.

   (e) (4%) The dynamic system $\mathbf{y}_{k+1} = A\mathbf{y}_k + \mathbf{y}_k$ is stable.

2. (40%) True or false. Give a reason or counterexample if false. All the following matrices are real.

   (a) (4%) Let A be a 3 × 3 matrix. If A has eigenvalues $0, 1, 1$, then $\text{rank}(A) = 2$.

   (b) (4%) Let A be a 3 × 3 matrix. If A has eigenvalues $0, 0, 1$, then $\text{rank}(A) = 1$.

   (c) (4%) Let A be a 3 × 3 matrix. If A has eigenvalues $0, 0, 0$, then $A^3 = 0$.

   (d) (4%) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is similar to $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

   (e) (4%) If A is similar to B and C, then A is similar to $B + C$.

   (f) (4%) Let $A = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b and c are real. If $a < 0$ and $c < 0$, then the solution of $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ approaches zero as $t \to \infty$.

   (g) (4%) Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. If $a > 0$ and $c > 0$, then A is positive definite.

   (h) (4%) If A is antisymmetric, i.e., $A^T = -A$, then A^2 is positive definite.

   (i) (4%) Let A be an $n \times n$ matrix. If $A = QQ^T$, where Q is an orthogonal matrix, then A is positive definite.

   (j) (4%) If A is symmetric positive definite, then A is invertible and A^{-1} is symmetric positive definite.

3. (40%) Answer the following questions.

(a) (8%) Suppose a particle is moving in a planar force filed and its position vector **x** satisfies $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ and $\mathbf{x}(0) = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$, where $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$. Solve the differential equation for $t \geq 0$.

(b) (8%) Find an invertible matrix S and a matrix of the form $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} = SBS^{-1}$.

(c) (8%) Let A and B be 3 × 3 matrices, and A is similar to B. Show that e^A is similar to e^B.

(d) (8%) Let A be a 2 × 2 matrix. The characteristic polynomial of A is $p(\lambda) = \lambda^2 - 2\lambda + 1$. Use the Cayley-Hamilton theorem to find a and b so that $A^3 = aA + bI$.

(e) (8%) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$. Find the maximum value of $\|A\mathbf{x}\|$, where **x** is a unit vector in $\mathbb{R}^2$.