Linear Algebra
Solutions to Final Exam 2009

1.

(a) P has eigenvalues 1, 0, 0. Let $\mathbf{v}=\begin{bmatrix}1\\1\\1\end{bmatrix}$, and then $P=\dfrac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$. Because

$P\mathbf{v}=\dfrac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\mathbf{v}=\mathbf{v}$, P has an eigenvalue 1. For any vector $\mathbf{u}$ in the orthogonal complement $L^{\perp}$, we have $P\mathbf{u}=\dfrac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\mathbf{u}=\mathbf{0}$, implying that P has repeated eigenvalues 0, 0.

(b) Because $Q = I\text{-}P$, $Q\mathbf{x}=(I-P)\mathbf{x}=\mathbf{x}-P\mathbf{x}=(1-\lambda)\mathbf{x}$. Thus, Q has eigenvalues 0, 1, 1.

(c) YES. Since P is a real symmetric matrix, P is orthogonally diagonalizable.

(d) Note that $L^{\perp}$ is the nullspace of $\begin{bmatrix}1 & 1 & 1\end{bmatrix}$. A basis vector for $L^{\perp}$ is

$\begin{bmatrix}-1\\1\\0\end{bmatrix}$. Then, consider the nullspace of $\begin{bmatrix}1 & 1 & 1\\-1 & 1 & 0\end{bmatrix}$. The basis of this

nullspace is $\begin{bmatrix}-1\\-1\\2\end{bmatrix}$. Therefore, an orthonormal basis for $L^{\perp}$ contains

vectors $\dfrac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix},\dfrac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$.

(e) The closest vector in the line L is $\mathbf{p}=P\begin{bmatrix}1\\2\\3\end{bmatrix}=\dfrac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$.

Then, the closest vector in $L^{\perp}$ is $(I-P)\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}-\begin{bmatrix}2\\2\\2\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

2.

(a) Since A has distinct eigenvalues, its eigenvectors are independent. Every $\mathbf{x} = c\mathbf{u}$ satisfies $A\mathbf{x} = \mathbf{0}$, and thus dim $N(A) = 1$.

(b) Note that $\det(A^2 - I) = \det(A-I)(A+I) = \det(A-I)\det(A+I)$, but $\det(A+I) = 0$. This is because -1 is an eigenvalue of A. Hence, $\det(A^2 - I) = 0$.

(c) The eigenvalues of A^2 are 0, 1, and 4. Then, trace(A^2) = 0 + 1 + 4 = 5.

(d) The eigenvalues of $(I-A)^{-1}$ are given by $(1-\lambda)^{-1}$, where λ's are eigenvalue of A, i.e., 1, 1/2, 1/3.

(e) Since **u**, **v**, and **w** are independent, the rank of S is 3.

(f) Any vector **x** in $\mathbf{R}^3$ can be expressed uniquely as $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$, and then $A\mathbf{x} = c_1A\mathbf{u} + c_2A\mathbf{v} + c_3A\mathbf{w} = -c_2\mathbf{v} - 2c_3\mathbf{w}$. A basis for the column space of A contains **v** and **w**.

(g) Suppose the eigen-equation is $A\mathbf{x} = \lambda\mathbf{x}$. Expanding $e^A = I + A + \dfrac{A}{2!} + \cdots$,

we have $e^A\mathbf{x} = \left(1 + \lambda + \dfrac{\lambda^2}{2!} + \cdots\right)\mathbf{x} = e^{\lambda}\mathbf{x}$. This concludes that the

eigenvalues of e^A are e^0, e^{-1}, e^{-2}.

(h) NO. For example, $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$. Since both A and B are

diagonalizable, there exists an invertible matrix M such that $B = MAM^{-1}$.

(i) NO. Every orthogonal matrix satisfies $A^T = A^{-1}$, requiring that

$\|A\mathbf{x}\| = \|\mathbf{x}\|$. Thus, the eigenvalues are $|\lambda| = 1$.

3.

(a) The given condition $Q^TQ = I$ indicates that the columns of Q form an orthonormal set. Hence, rank$Q = n$.

(b) Consider the normal equation $Q^TQ\hat{\mathbf{x}} = Q^T\mathbf{b}$. The projection of **b** onto the column space of Q is computed by $P\mathbf{b} = Q\hat{\mathbf{x}} = Q(Q^TQ)^{-1}Q^T\mathbf{b} = QQ^T\mathbf{b}$, and thus the projection matrix is QQ^T.

(c) When det(QQ^T) = 0, rank(QQ^T) < m. However, rank(QQ^T) = rank(Q^T) = n. Therefore, we are sure that $n < m$.

4.

(a) $\det A = \begin{vmatrix} c & 2 & 1 \\ 2 & c & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2(c-2)(c+1)$ has roots $c = 2, -1$.

(b) We can use the determinants of left upper submatrices to check. For the 1 by 1 submatrix, we require $c > 0$. For the 2 by 2 submatrix, we require $c^2 - 4 > 0$, and thus $c > 2$. For A itself, $2(c-2)(c+1) > 0$ if $c > 2$. Therefore, $c > 2$ ensures that A is positive definite.

(c) First note that every symmetric matrix has real eigenvalues. To ensure that the exponents e^{At} converging to the zero matrix, all the eigenvalues must be negative, implying that $-A$ is positive definite. For the 1 by 1 submatrix, we require $c < 0$. For the 2 by 2 submatrix, we require $c^2 - 4 > 0$, and thus $c > 2$ or $c < -2$. For $-A$. itself, $2(c-2)(c+1) < 0$, or $-1 < c < 2$. So there is no such c making A negative definite.

5.

(a) From $\mathbf{u}_k = A^k \mathbf{u}_0 = S\Lambda^k S^{-1}\mathbf{u}_0$, we can obtain

$\mathbf{u}_k = c_1\lambda_1^k \mathbf{x}_1 + c_2\lambda_2^k \mathbf{x}_2 + c_3\lambda_3^k \mathbf{x}_3$, where the coefficient vector is given by

$\mathbf{c} = S^{-1}\mathbf{u}_0$.

(b) $\lambda_1 = 1, |\lambda_2| < 1, |\lambda_3| < 1$.

(c) The conditions $|e^{\lambda_i}| < 1$ are equivalent to $\lambda_i < 0$ for every i.

6.

(a) Since A is already in the diagonal form, we know immediate that the eigenvalues are 3 and 7, the corresponding eigenvectors are $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

(b) $\|A\mathbf{x}\|$ is maximized at $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, and the maximum value of it is 7.

(c) A is diagonalizable alright. Since A has nonzero eigenvalues, it is invertible. Because A has the form $A = Q\Lambda Q^T$, A is clearly symmetric. Since A has eigenvalues 3 and 7, it is neither an orthogonal matrix nor a projection matrix.