Linear Algebra
Solutions to Final Exam 2012

1.

(a) Since $A\mathbf{u}=\mathbf{0}$, the nullspace of A is spanned by **u**. Thus, $\dim N(A)=1$. By the rank-nullity theorem, $\text{rank}(A)=n-\dim N(A)=3-1=2$. Note that the column space of A is spanned by **v** and **w**.

(b) Note that the eigenvalues of A^3 are 0, -8, 27. The trace of A^3 is the sum of its eigenvalues. Hence, $\text{trace}(A^3)=0-8+27=19$.

(c) $\det(A+I)^{-1}=\dfrac{1}{\det(A+I)}=\dfrac{1}{(0+1)(-2+1)(3+1)}=-\dfrac{1}{4}$

(d) Every real projection matrix P satisfies $P^T=P$ and $P^2=P$, implying that the eigenvalues of P are real and $\lambda^2=\lambda$. So, the eigenvalues can be either 0 or 1. Because $C(P)=C(A)$, it follows that $\dim C(P)=2$. Therefore, P has eigenvalues 0, 1, 1.

(e) Since A has distinct eigenvalues, the corresponding eigenvectors **v**, **u**, **w** are linearly independent. Any vector $\mathbf{x}\in\mathbb{R}^3$ can be expressed as $\mathbf{x}=c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w}$ in a unique way. The linear equation becomes $A\mathbf{x}=-2c_2\mathbf{v}+3c_3\mathbf{w}=\mathbf{v}+\mathbf{w}$. A particular solution is $\mathbf{x}_p=-\dfrac{1}{2}\mathbf{v}+\dfrac{1}{3}\mathbf{w}$. The general solutions are $\mathbf{x}=\mathbf{x}_p+c\mathbf{u}=-\dfrac{1}{2}\mathbf{v}+\dfrac{1}{3}\mathbf{w}+c\mathbf{u}$, where c is a scalar.

(f) TRUE. Note that A has distinct eigenvalues. This means that A is diagonalizable and similar to $\text{diag}(0,-2,3)$. Because B also has distinct eigenvalues $0,-2$, 3, B is similar to $\text{diag}(0,-2,3)$. This concludes that A is similar to B.

(g) TRUE. Because **u**, **v**, **w** are linearly independent, **u** is not in the span of **v** and **w**, implying that **u** is not in the column space of A.

(h) FALSE. For example, $A=\begin{bmatrix}0&1&0\\0&-2&0\\0&0&3\end{bmatrix}$. Apparently, $C(A^T)\neq C(A)$.

(i) TRUE. From (d), every real symmetric matrix is orthogonally diagonalizable.

(j) FALSE. As the example in (h), $A^TA=\begin{bmatrix}0&0&0\\0&5&0\\0&0&9\end{bmatrix}$ has eigenvalues 0, 5, 9.

2.

(a) Diagonalize A as $A = Q\Lambda Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}9 & 0\\ 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Since $(\pm 3)^2 = 9,\ (\pm 1)^2 = 1$, there are four square roots. One is $R = Q\Lambda^{1/2}Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$. And other three square roots are $\begin{bmatrix}-2 & -1\\ -1 & -2\end{bmatrix}, \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}, \begin{bmatrix}-1 & -2\\ -2 & -1\end{bmatrix}$.

(b) $e^A = Se^{\Lambda}S^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^9 & 0\\ 0 & e\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e^9+e & e^9-e\\ e^9-e & e^9+e\end{bmatrix}$.

(c) The characteristic polynomial of A is $\det(A-\lambda I) = \lambda^2 - 10\lambda + 9$. According to Cayley-Hamilton theorem, $A^2 - 10A + 9I = 0$. Thus,

$$A^3 = A(A^2) = A(10A - 9I) = 10A^2 - 9A$$
$$= 10(10A - 9I) - 9A = 91A - 90I.$$

(d) Note that $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T Q\Lambda Q^T\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y}$, where $\mathbf{y} = Q^T\mathbf{x}$. Because $\|\mathbf{y}\| = \|\mathbf{x}\| = 1$, $\mathbf{x}^T A\mathbf{x}$ is maximized when $\mathbf{y} = \begin{bmatrix}1\\ 0\end{bmatrix}$, i.e., $\mathbf{x} = Q\mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$. The maximum value of $\mathbf{x}^T A\mathbf{x}$ is 9.

3.

(a) A has eigenvalue $\lambda_1 = a - bi = 1 - 3i$ with eigenvector $\mathbf{x}_1 = \begin{bmatrix}1+3i\\ 2\end{bmatrix}$. Hence, $C = \begin{bmatrix}1 & -3\\ 3 & 1\end{bmatrix}$ and $S = \begin{bmatrix}\operatorname{Re}\mathbf{x}_1 & \operatorname{Im}\mathbf{x}_1\end{bmatrix} = \begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}$. Note that S and C are not unique.

(b) $\|C\mathbf{x}\| = \|\mathbf{x}\|$ implies $\|C\mathbf{x}\|^2 = (C\mathbf{x})^T(C\mathbf{x}) = \mathbf{x}^T C^T C\mathbf{x} = \|\mathbf{x}\|^2$. Thus, $C^T C = I$. C is an orthogonal matrix, so $a^2 + b^2 = 1$.

(c) Note that C has eigenvalues $\lambda = a \pm bi$. To ensure C is stable, we require $|\lambda| = |a \pm bi| = \sqrt{a^2+b^2} < 1$. This is equivalent to $a^2 + b^2 < 1$. Another reasoning is as follows. Write $C = \begin{bmatrix}a & -b\\ b & a\end{bmatrix} = r\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$. As $k \to \infty$,

$$C^k = r^k\begin{bmatrix}\cos k\theta & -\sin k\theta\\ \sin k\theta & \cos k\theta\end{bmatrix} \to 0 \quad \text{if} \quad r = \sqrt{a^2+b^2} < 1.$$

4.

(a) Note that $E\mathbf{e} = (\mathbf{e}\mathbf{e}^T)\mathbf{e} = 3\mathbf{e}$, indicating that E has eigenvalue 3 and eigenvector $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$. Since E is a rank-one matrix, the other two eigenvalues must be zero. Because E is symmetric, all eigenvectors can be chosen to be orthonormal. By inspection, the two eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix} \text{ and } \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}.$$

(b) Suppose $E\mathbf{x} = \lambda\mathbf{x}$. Then, $A\mathbf{x} = (c-1)I\mathbf{x} + E\mathbf{x} = (c-1+\lambda)\mathbf{x}$, and thus A has eigenvalues $c+2, c-1, c-1,$ and the corresponding orthonormal eigenvectors are $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}$.

(c) To ensure A is nonsingular, we require $c \neq -2$ and $c \neq 1$. Since $E^2 = 3E$, the inverse A must have the form of $A^{-1} = aI + bE$. Then, $A^{-1}A = (aI + bE)((c-1)I + E) = (ac - a)I + (a + bc - b + 3b)E$, and thus $ac - a = 1,\ a + bc + 2b = 0$. Solving yields $a = \frac{1}{c-1}$ and $b = \frac{1}{(c+2)(1-c)}$.

Therefore, $A^{-1} = \frac{1}{c-1}I + \frac{1}{(c+2)(1-c)}E$.

5.

(a) Let $A = MBM^{-1}$. Then,
$A + I = MBM^{-1} + I = MBM^{-1} + MIM^{-1} = M(B+I)M^{-1}$.

(b) $AB = ABAA^{-1} = A(BA)A^{-1}$.

(c) Note that $\mathbf{x}^T A\mathbf{x} = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x} = -\mathbf{x}^T A\mathbf{x}$. So, $\mathbf{x}^T A\mathbf{x} = 0$ for every real vector $\mathbf{x}$.