Linear Algebra
Solutions to Final Exam 2013

1.

(a) Note that the eigenvalues of A^2 are $\alpha^2,\ \beta^2,\ \gamma^2$. But $A^2 = 0$ implies that all the eigenvalues are zero. Therefore, $\alpha = \beta = \gamma = 0$.

(b) Because $\det(A^2) = \alpha^2\beta^2\gamma^2 = 0$, we know that at least one eigenvalue is zero.

(c) If *A* has distinct eigenvalues, then *A* is diagonalizable. Thus, if *A* is not diagonalizable, then *A* has repeated eigenvalues. This means that *A* has at most two distinct eigenvalues.

(d) Symmetric positive definite matrices have positive eigenvalues. So, $\alpha, \beta, \gamma > 0$.

(e) Suppose $A\mathbf{x} = \lambda\mathbf{x}$. It follows that $(A - I)\mathbf{x} = \lambda\mathbf{x} - \mathbf{x} = (\lambda - 1)\mathbf{x}$. Therefore, *A* has eigenvalues 2, 6, 7.

(f) Two similar matrices have the same eigenvalues. Note that the characteristic polynomial of $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & -1 & 5 \end{bmatrix}$

$p(\lambda) = (5-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)(4-\lambda)$. Therefore, *A* has eigenvalues 2, 4, 5.

(g) Orthogonal projection matrix *A* satisfies $A^2 = A = A^T$. Let λ be an eigenvalue of *A*. From (a), it follows that $\lambda^2 = \lambda$, implying $\lambda = 0$ or $\lambda = 1$. Thus, $\alpha, \beta, \gamma \in \{0, 1\}$.

(h) Orthogonal matrix *A* has the property $A^T = A^{-1}$, meaning that $\|A\mathbf{x}\|^2 = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$ for every $\mathbf{x}$. Thus, $|\alpha| = |\beta| = |\gamma| = 1$.

(i) Symmetric matrices are orthogonally diagonalizable, implying that the eigenvectors are linearly independent. Since $C(A) = \text{span}(\mathbf{u}, \mathbf{v})$, it follows that $A\mathbf{x} = A(c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}) = c_1\alpha\mathbf{u} + c_2\beta\mathbf{v}$. So, we know that $\alpha\beta \neq 0,\ \gamma = 0$.

(j) $(A+I)^k$ approaches the zero matrix as $k \to \infty$ if and only if $|\lambda(A+I)| < 1$, i.e., $|\lambda(A) + 1| < 1$. Therefore, $-2 < \alpha, \beta, \gamma < 0$.

2.

(a) Let $B=\begin{bmatrix}1 & 3\\ 1 & 2\end{bmatrix}$. The matrix representation of *A* relative to the basis

$$\beta=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}3\\2\end{bmatrix}\right\}$$ is computed by

$$B^{-1}AB=\begin{bmatrix}1 & 3\\ 1 & 2\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 2\end{bmatrix}.$$

(b) Since *A* is invertible, we have $A^{-1}(AB)A=BA$. It concludes that *AB* is similar to *BA*, for every *B*.

(c) If *AB* is symmetric, then $(AB)^T=B^TA^T=BA=AB$. Then,
$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a & b\\ b & c\end{bmatrix}=\begin{bmatrix}a & b\\ b & c\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$. Comparing terms yields $a=b+c$.

(d) Let $B=\begin{bmatrix}I & A\\ A & I\end{bmatrix}=\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}$. Test the determinants of the leading

principal submatrices: $|B_1|=|1|=1$, $|B_2|=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1$, $|B_3|=\begin{vmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{vmatrix}=-1$.

This indicates that *B* is not positive definite.

(e) There are many possibilities. By inspection, we have
$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$. So, $M=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

(f) Note that *A* can be orthogonally diagonalized as

$$A=Q\Lambda Q^T=[\mathbf{q}_1\quad \mathbf{q}_2]\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}\begin{bmatrix}\mathbf{q}_1^T\\ \mathbf{q}_2^T\end{bmatrix}=\lambda_1\mathbf{q}_1\mathbf{q}_1^T+\lambda_2\mathbf{q}_2\mathbf{q}_2^T=B+C,$$ where

$Q^T=Q^{-1}$. Since *Q* is an orthogonal matrix, we have

$BC=\left(\lambda_1\mathbf{q}_1\mathbf{q}_1^T\right)\left(\lambda_2\mathbf{q}_2\mathbf{q}_2^T\right)=\lambda_1\lambda_2\mathbf{q}_1\mathbf{q}_1^T\mathbf{q}_2\mathbf{q}_2^T=0$. It is easy to find the

eigenvalues and orthonormal eigenvectors: $\lambda_1=\dfrac{1+\sqrt{5}}{2}$, $\lambda_2=\dfrac{1-\sqrt{5}}{2}$, and

$\mathbf{q}_1=\dfrac{1}{\sqrt{\lambda_1^2+1}}\begin{bmatrix}\lambda_1\\ 1\end{bmatrix}$, $\mathbf{q}_2=\dfrac{1}{\sqrt{\lambda_2^2+1}}\begin{bmatrix}\lambda_2\\ 1\end{bmatrix}$. Also note that

$\lambda_1+\lambda_2=1,\ \lambda_1\lambda_2=-1.$ Thus,

$$B=\frac{\lambda_1}{\lambda_1^2+1}\begin{bmatrix}\lambda_1^2 & \lambda_1\\ \lambda_1 & 1\end{bmatrix}=\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1^2 & \lambda_1\\ \lambda_1 & 1\end{bmatrix},$$

$$C=\frac{\lambda_2}{\lambda_2^2+1}\begin{bmatrix}\lambda_2^2 & \lambda_2\\ \lambda_2 & 1\end{bmatrix}=\frac{1}{\lambda_2-\lambda_1}\begin{bmatrix}\lambda_2^2 & \lambda_2\\ \lambda_2 & 1\end{bmatrix}.$$

(g) Since the eigenvalues of matrix exponential e^A are $e^{\lambda_1}, e^{\lambda_2}$, we obtain $\det(e^A)=e^{\lambda_1}e^{\lambda_2}=e^{\lambda_1+\lambda_2}=e^{\mathrm{tr}(A)}=e.$

(h) Note that $e^{Bt}\to 0$ as $t\to\infty$ if and only if the real part of every eigenvalue of B is negative. Thus, we require $\lambda_1+k<0$ and $\lambda_2+k<0$, which is equivalent to $k<-\frac{1+\sqrt{5}}{2}$.

(i) Note that $\mathbf{x}^TA\mathbf{x}=\mathbf{x}^TQ\Lambda Q^T\mathbf{x}=\mathbf{y}^T\Lambda\mathbf{y}$, where $\mathbf{y}=Q^T\mathbf{x}$. Because $\|\mathbf{y}\|=\|\mathbf{x}\|=1$, $\mathbf{x}^TA\mathbf{x}$ is minimized when $\mathbf{y}=\begin{bmatrix}0\\1\end{bmatrix}$, i.e.,

$$\mathbf{x}=Q\mathbf{y}=\mathbf{q}_2=\frac{1}{\sqrt{\lambda_2^2+1}}\begin{bmatrix}\lambda_2\\1\end{bmatrix}.$$ The minimum value of $\mathbf{x}^TA\mathbf{x}$ is

$$\lambda_2=\frac{1-\sqrt{5}}{2}.$$

(j) We observe that AB is an orthogonal matrix. An apparent choice of B is $B=A^{-1}Q$, where Q is an orthogonal matrix. That is,

$$B=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta-\sin\theta & -\sin\theta-\cos\theta\end{bmatrix}$$ or

$$B=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}=\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta-\sin\theta & \sin\theta+\cos\theta\end{bmatrix}$$ will do. For example, $B=\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}$ is a nonsymmetric matrix.

3.

(a) Clearly, λ is an eigenvalue of A. Thus, $\det(A^T-\lambda I)=\det(A-\lambda I)=0$, implying that $A^T-\lambda I$ is singular.

(b) If A is similar to B, then there exists a nonsingular matrix M so that $B=M^{-1}AM$. Thus, $B^{-1}=(M^{-1}AM)^{-1}=M^{-1}A^{-1}M$. This concludes that A^{-1} is similar to B^{-1}.

(c) If A is real symmetric, then A is orthogonally diagonalizable, i.e.,

$A = Q\Lambda Q^T$. Then, $e^A = Qe^{\Lambda}Q^T$, where $e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$ has positive diagonal entries, i.e., positive eigenvalues. It is easy to see that e^A is symmetric.

(d) Cayley-Hamilton theorem states that the characteristic polynomial of A annihilates itself. So $p(\lambda) = \lambda^3 + \lambda^2 - 2\lambda = \lambda(\lambda - 1)(\lambda + 2)$. This shows that A has eigenvalues $0, 1, -2$.