**Linear Algebra**
**Solutions to Final Exam 2015**

1.

(a) If $A\mathbf{x}=\lambda\mathbf{x}$, then $(2A-3I)\mathbf{x}=(2\lambda-3)\mathbf{x}$. By solving $2\lambda-3=1,5,7$, respectively, the eigenvalues of $A$ are 2. 4, 5.

(b) According to Cayley-Hamilton theorem, the characteristic polynomial of $A$ is $p(\lambda)=\lambda^3-\lambda=\lambda(\lambda-1)(\lambda+1)$. Therefore, the eigenvalues of $A$ are $0,1,-1$.

(c) Symmetric matrices have real eigenvalues, $\alpha,\beta,\gamma\in\mathbb{R}$.

(d) Because $\det(A)=\alpha\beta\gamma=0$ and $\operatorname{trace}(A)=\alpha+\beta+\gamma>0$, we know that at least one eigenvalue is zero, and the sum of nonzero eigenvalues is positive.

(e) Note that the dimension of the nullspace of $A$ is 2, meaning that there are two linearly independent eigenvectors corresponding to the zero eigenvalue. Since $A$ is not diagonalizable, it follows that $\alpha=\beta=\gamma=0$.

(f) According to the rank-nullity theorem, $\dim N(A)=3-\operatorname{rank}(A)=2$. This means that $A$ has at least two zero eigenvalues.

(g) Because $A$ is similar to $B$, they have the same eigenvalues. Hence, the eigenvalues of $A$ are $-1,-1,2$.

(h) Permutation matrix $A$ satisfies $\|A\mathbf{x}\|=\|\mathbf{x}\|$, for every $\mathbf{x}$. Hence,

$|\alpha|=|\beta|=|\gamma|=1$.

(i) Since $A$ is an orthogonal projection matrix, we have $A^2=A$ and $A^T=A$, implying that $A$ is diagonalizable and the eigenvalues are either 0 or 1. Because $\operatorname{rank}(A)=\dim C(A)=2$, it follows that $\alpha=\beta=1$ and $\gamma=0$.

(j) $\mathbf{y}(t)$ approaches the zero vector as $t\to\infty$ if and only if every eigenvalue $\lambda$ satisfies $\operatorname{Re}(\lambda)<0$.

2.

(a) FALSE. For example, $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, but $A^2=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$.

(b) FALSE. If $A$ has eigenvalues 0,0,1, then $\dim N(A)=3-\operatorname{rank}(A)\geq 1$, or

$\operatorname{rank}(A)\leq 2$. For example, the rank of $\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$ is 1, and the rank of

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 2.

(c) TRUE. If $A = SDS^{-1}$, where $D$ is diagonal, then $I + A + A^2 = SIS^{-1} + SDS^{-1} + SD^2S^{-1} = S(I + D + D^2)S^{-1}$, where $I + D + D^2$ is diagonal.

(d) TRUE. Since $A$ is invertible, we have $A^{-1}(AB + I)A = BA + I$.

(e) FALSE. For example, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not similar to $B = 0$. However, $A^2 = B^2 = 0$.

(f) TRUE. If $A$ has real eigenvalues and orthogonal eigenvectors, then $A$ can be expressed in the form of $A = QDQ^T$, where $Q^T = Q^{-1}$ and $D$ is real and diagonal. Thus, $A^T = (QDQ^T)^T = QD^TQ^T = QDQ^T = A$.

(g) TRUE. If $A$ is $n$ by $n$, then $\det A = \det A^T = \det(-A) = (-1)^n \det A$. If $n$ is an odd number, then $\det A = 0$. Since $A$ is nonsingular, $n$ must be an even number, and all eigenvalues of $A$ are nonzero pure imaginary conjugates $\pm bi$. This proves $\det A > 0$.

(h) FALSE. For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, but

$$B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}$$ is not symmetric.

(i) TRUE. Note that $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$

(j) TRUE. It can be seen from $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} d & c \\ b & a \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3.

(a) Diagonalize the given two matrices as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = SDS^{-1}$$ and

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = TDT^{-1}.$$ Thus,

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = ST^{-1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}TS^{-1}$. It follows that $TS^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Hence,

$M = c\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $c$ is any nonzero number.

(b) Note that $A = \begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 3-2i$ and $\lambda_2 = 3+2i$, and the corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2+2i \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2-2i \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. It can be easily seen that

$\begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}^{-1}$. Hence, $M = c\begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$, where $c$ is any nonzero number.

(c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ and the corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Write

$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$. Therefore,

$$\cos A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \cos(-1) & 0 \\ 0 & \cos(1) \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \cos(1)I.$$

(d) The maximum value of $\|A\mathbf{x}\|^2 = \mathbf{x}^T A^T A\mathbf{x}$ subject to $\|\mathbf{x}\| = 1$ is the largest eigenvalue of $A^T A$. Note that the eigenvalues of $A^T A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$ are 6, 4, 0. Therefore, $\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \sqrt{6}$.

(e) The characteristic polynomial of $A$ is $p(t) = (t-1)^2(t-2)$.
Cayley-Hamilton theorem states that $(A-I)^2(A-2I) = A^3 - 4A^2 + 5A - 2I$.
Multiplying $A^{-1}$ yields $A^{-1} = \frac{1}{2}A^2 - 2A + \frac{5}{2}I$. So, $a = \frac{1}{2}, b = -2, c = \frac{5}{2}$.