Linear Algebra
Solutions to Final Exam 2016

1.

(a) Since matrix A is real symmetric, it is orthogonally diagonalizable.

Consider $A = Q\Lambda Q^T$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $Q^T = Q^{-1}$. Then,

$\text{rank}(A) = \text{rank}(Q\Lambda Q^T) = \text{rank}(\Lambda) = 2$, we can conclude that A has one zero eigenvalue and two nonzero eigenvalues.

(b) Suppose A is the reflection matrix about the plane spanned by **u** and **v**, it follows that $A\mathbf{u} = \mathbf{u}$, $A\mathbf{v} = \mathbf{v}$, and $A\mathbf{w} = -\mathbf{w}$, where **w** is the normal vector of the plane. Thus, the eigenvalues of A are $1, 1, -1$.

(c) Suppose A is the projection matrix onto the plane spanned by **u** and **v**, it follows that $A\mathbf{u} = \mathbf{u}$, $A\mathbf{v} = \mathbf{v}$, and $A\mathbf{w} = \mathbf{0}$, where **w** is the normal vector of the plane. Thus, the eigenvalues of A are $1, 1, 0$.

(d) From (a), $(A^2 + A - 2I) = QD^2Q^T + QDQ^T - 2I = Q(D^2 + D - 2I)Q^T = 0$. Thus, $D^2 + D - 2I = 0$. So the eigenvalues of A satisfy $\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$. But $\text{trace}(A) = 0$, we can conclude that the eigenvalues of A are $1, 1, -2$.

(e) Consider $\mathbf{y}_{k+1} = (A + I)\mathbf{y}_k$. Note that $\mathbf{y}_k$ approaches the zero vector as

$k \to \infty$ if and only if every eigenvalue μ of $A + I$ satisfies $|\mu| < 1$.

Thus, every eigenvalue λ of A satisfies $|\lambda + 1| < 1$.

2.

(a) TRUE. Since $\dim N(A) = 1$, the rank-nullity theorem says that $\text{rank}(A) = 3 - \dim N(A) = 2$.

(b) FALSE. For example, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) TRUE. The characteristic polynomial of A is $p(\lambda) = \lambda^3$. By the Cayley-Hamilton theorem, $A^3 = 0$.

(d) TRUE. Note that $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ have distinct eigenvalues 1 and 3.

Thus, both matrices are similar to $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

(e) FALSE. For example, $A=\begin{bmatrix}1&0\\0&2\end{bmatrix}$, $B=\begin{bmatrix}1&1\\0&2\end{bmatrix}$ and $C=\begin{bmatrix}1&2\\0&2\end{bmatrix}$.

(f) TRUE. $A=\begin{bmatrix}a&-b&0\\b&a&0\\0&0&c\end{bmatrix}$ has eigenvalues $a\pm bi$ and c. To ensure a stable system, it is required that $a<0$ and $c<0$.

(g) FALSE. For example, $\begin{bmatrix}1&2\\2&4\end{bmatrix}$ has eigenvalues 0 and 5 and thus is not positive definite.

(h) FALSE. Note that $(A^2)^T = A^TA^T = (-A)(-A)=A^2$. Hence, A^2 is symmetric. However, A^2 is not necessarily positive definite. For example, $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$.

(i) TRUE. Note that $A=QIQ^T$. It is clear that the eigenvalues of A are 1 and thus it is positive definite.

(j) TRUE. If A is symmetric positive definite, then A can be expressed in the form of $A=Q\Lambda Q^T$, where $Q^T=Q^{-1}$ and Λ is a diagonal matrix with positive diagonal entries. Thus, $A^{-1}=(Q\Lambda Q^T)^{-1}=Q\Lambda^{-1}Q^T$ is symmetric positive definite.

3.

(a) The eigenvalues of A turn out to be $\lambda_1=6$ and $\lambda_2=-1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}-5\\2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. The solution is given by $\mathbf{x}(t)=c_1e^{6t}\begin{bmatrix}-5\\2\end{bmatrix}+c_2e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$. We want c_1 and c_2 to satisfy $\mathbf{x}(0)=c_1\begin{bmatrix}-5\\2\end{bmatrix}+c_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}13\\-1\end{bmatrix}$. Calculations show that $c_1=-2$ and $c_2=3$. Thus, $\mathbf{x}(t)=-2e^{6t}\begin{bmatrix}-5\\2\end{bmatrix}+3e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$.

(b) Note that $A=\begin{bmatrix}1&5\\-2&3\end{bmatrix}$ has eigenvalues $\lambda_1=2-3i$ and $\lambda_2=2+3i$, and the corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}5\\1-3i\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}+i\begin{bmatrix}0\\-3\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}5\\1+3i\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}+i\begin{bmatrix}0\\3\end{bmatrix}$. It is seen that

$\begin{bmatrix}1&5\\-2&3\end{bmatrix}=\begin{bmatrix}5&0\\1&-3\end{bmatrix}\begin{bmatrix}2&-3\\3&2\end{bmatrix}\begin{bmatrix}5&0\\1&-3\end{bmatrix}^{-1}$. Hence, $B=\begin{bmatrix}2&-3\\3&2\end{bmatrix}$ and

$S = k\begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix}$, where k is any nonzero number.

(c) Let $A = SBS^{-1}$. Then,

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2} + \cdots \\ &= SIS^{-1} + SBS^{-1} + \frac{SB^2S^{-1}}{2} + \cdots \\ &= S\left(I + B + \frac{B^2}{2} + \cdots \right) S^{-1} \\ &= Se^B S^{-1}. \end{aligned}$$

(d) The Cayley-Hamilton theorem states that $A^2 - 2A + I = 0$. That is, $A^2 = 2A - I$. Thus,

$$\begin{aligned} A^3 &= A(A^2) = A(2A - I) = 2A^2 - A \\ &= 2(2A - I) - A = 3A - 2I \end{aligned}.$$

(e) The maximum value of $\|A\mathbf{x}\|^2 = \mathbf{x}^T A^T A\mathbf{x}$ subject to $\|\mathbf{x}\| = 1$ is the largest eigenvalue of $A^T A$. Note that the eigenvalues of $A^T A = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are 11 and 1.

Therefore, $\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \sqrt{11}$.