Linear Algebra
Solutions to Quiz 2 2005

1.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then, we know that the nullspace matrix is $N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. The problem is find the least-squares solution to $N\mathbf{y}=\mathbf{b}$, which requires $N^T(\mathbf{b} - N\hat{\mathbf{y}}) = 0$. The normal equation is $N^T N\hat{\mathbf{y}} = N^T\mathbf{b}$. Since $N = [\mathbf{n}_1 \quad \mathbf{n}_2]$ has orthogonal columns, $N^T N = \begin{bmatrix} \|\mathbf{n}_1\|^2 & 0 \\ 0 & \|\mathbf{n}_2\|^2 \end{bmatrix}$. Hence, the projected vector of $\mathbf{b}$ on $N(A)$ is

$$\mathbf{p} = N\hat{\mathbf{y}} = \frac{\mathbf{b}^T\mathbf{n}_1}{\|\mathbf{n}_1\|^2}\mathbf{n}_1 + \frac{\mathbf{b}^T\mathbf{n}_2}{\|\mathbf{n}_2\|^2}\mathbf{n}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

(b) The columns of the nullspace matrix are already orthogonal. We only need to normalize them: $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) From problem (a), the row space of A is spanned by two orthogonal vectors, i.e., $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. The projection matrix P is thus

$$P = \frac{\mathbf{u}_1\mathbf{u}_1^T}{\|\mathbf{u}_1\|^2} + \frac{\mathbf{u}_2\mathbf{u}_2^T}{\|\mathbf{u}_2\|^2} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(d) With the result in (a), we can obtain the required vector directly by $\mathbf{p}' = \mathbf{b} - \mathbf{p} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(e) Since $P^2=P$, we have det P^2=(det $P)^2$=det P. Solving this equation yields det P=0 or 1.

2.

(a) FALSE. For example, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, but $QQ^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I_4$.

(b) TRUE. Let $\mathbf{y}=A\mathbf{x}$. Then $\mathbf{y}$ is in $C(A)$. But $A^TA\mathbf{x}=A^T\mathbf{y}=\mathbf{0}$ implies that $\mathbf{y}$ is also in $N(A^T)$. Since $C(A)$ is the orthogonal complement of $N(A^T)$, we have $\mathbf{y} \in C(A) \cap N(A^T) = \{\mathbf{0}\}$, and thus $\mathbf{y}=A\mathbf{x}=\mathbf{0}$.
Here is another proof. Multiply $\mathbf{x}^T$ to both sides of $A^TA\mathbf{x}=\mathbf{0}$, and then
$\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T A\mathbf{x} = \|A\mathbf{x}\|^2 = 0$, implying that $A\mathbf{x}=\mathbf{0}$.

(c) FALSE. If A is m by n, then A^TA is n by n. Clearly, $C(A) \subset \mathbf{R}^m$, but $C(A^TA) \subset \mathbf{R}^n$.

(d) TRUE. Since A is invertible and B is not invertible, we know that det B=0, and thus det C=det AB=(det A)(det B)=0. This concludes that C is singular (not invertible).

3.

(a) Because A is an upper triangular matrix, $\det A = 1$. It is easy to obtain the cofactor matrix and $A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) YES. Suppose A is an upper triangular matrix. Examining the entries of C (the cofactor matrix of A) will reveal the following property that C_{ij}=0 for every $i<j$.

(c) From $A^{-1} = \dfrac{C^T}{\det A}$, take determinant on both sides,

$$\det A^{-1} = \frac{1}{(\det A)^n} \det C^T = \frac{1}{(\det A)^n} \det C = \frac{1}{\det A}$$

Hence, det C=(det $A)^{n-1}$.